# Physics 443, Solutions to PS 4

1. **Neutrino Oscillations**

   (a) Energy eigenvalues and eigenvectors

   The eigenvalues of $H$ are $E_1 = E_0 + A$, and $E_2 = E_0 - A$ and the eigenvectors are

   $$\vec{\nu_1} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \quad \vec{\nu_2} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

   And

   $$\langle \nu_1 \mid \nu_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = 0$$

   (b) Similarity transformation

   $$S = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}$$

   $$\begin{aligned} S\mid \nu_e\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \\ S\mid \nu_\mu\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\end{aligned}$$

   (c)

   $$\mid \psi(t)\rangle = a'\mid \nu_1\rangle e^{-\frac{i}{\hbar}E_1 t} + b'\mid \nu_2\rangle e^{-\frac{i}{\hbar}E_2 t}$$

   (d) Time evolution

   At $t = 0$

   $$\begin{aligned}\langle \nu_e \mid \psi\rangle &= 1 \Rightarrow a'\langle \nu_e \mid \nu_1\rangle + b'\langle \nu_e \mid \nu_2\rangle = 1 \\ \langle \nu_\mu \mid \psi\rangle &= 0 \Rightarrow a'\langle \nu_\mu \mid \nu_1\rangle + b'\langle \nu_e \mid \nu_\mu\rangle = 0\end{aligned}$$

   The inner product

   $$\langle \nu_e \mid \nu_1\rangle = \begin{pmatrix}1 & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}$$

   Also $\langle \nu_e \mid \nu_2\rangle = \langle \nu_\mu \mid \nu_1\rangle = -\langle \nu_\mu \mid \nu_2\rangle = \frac{1}{\sqrt{2}}$. Substitution into the above gives $a' = b' = \frac{1}{\sqrt{2}}$. Therefore

   $$\mid \psi(t)\rangle = \frac{1}{\sqrt{2}}\left(\mid \nu_1\rangle e^{-\frac{i}{\hbar}E_1 t} + \mid \nu_2\rangle e^{-\frac{i}{\hbar}E_2 t}\right)$$

$$P_\mu(t) = |\langle \nu_\mu \mid \psi(t)\rangle|^2 = \sin^2 \frac{E_1 - E_2}{\hbar} t$$

$$P_e(t) = |\langle \nu_e \mid \psi(t)\rangle|^2 = \cos^2 \frac{E_1 - E_2}{\hbar} t$$

We would measure energy $E_1$ with probablity 1/2 and $E_2$ with probability 1/2.

2. **Griffiths 8.1**. The simplicity of the WKB method is that we can directly right down the form of the wavefunction, and the relevant quantization conditions imposed by the form of the boundary. In this case, we have that the WKB wavefunction is

$$\psi(x) \approx \frac{C}{\sqrt{P(x)}} \exp\left(\pm\frac{i}{\hbar}\int P(x)dx\right),$$

here $P(x) = \sqrt{2m(\epsilon - V(x))}$. Our connection formula show that the boundary conditions imply that our phase is quantized as

$$\int_0^a P(x)dx = \pi n\hbar,$$

where $n$ is an integer. Therefore,

$$\int_0^{a/2} \sqrt{2m(\epsilon - V_0)}\, dx + \int_{a/2}^a \sqrt{2m\epsilon}\, dx = \pi n\hbar,$$

$$\sqrt{2m\epsilon(1 - \frac{V_0}{\epsilon})} + \sqrt{2m\epsilon} = \pi n\hbar\frac{2}{a},$$

$$\epsilon\left(\sqrt{1 - \frac{V_0}{\epsilon}} + 1\right)^2 = \frac{4\pi^2 n^2\hbar^2}{2ma^2} = 4E_n^0,$$

$$\epsilon_n = \frac{E_n^0}{\left[\frac{1}{2}\left(\sqrt{1 - \frac{V_0}{\epsilon_n}} + 1\right)\right]^2} \approx \frac{E_n^0}{\left[\frac{1}{2}\left(\sqrt{1 - \frac{V_0}{E_n^0}} + 1\right)\right]^2}.$$

In order to compare with perturbation theory, we need to make the further assumption that $E_n \gg V_0$, which would then give that $\epsilon_n \approx E_n^0 + V_0/2$.

3. **Griffiths 8.6**. Analyze the bouncing ball (gravitational potential) problem using the WKB approximation.

(a) Find the allowed energies, $E_n$, in therms of $m, g$, and $h$. [The potential

$$V(x) = \begin{cases} 0 & x < 0 \\ mgx & x > 0 \end{cases} \tag{1}$$

We use the quantization condition for a potential well with one vertical wall, namely

$$\int_0^{x_t} p dx = \left(n - \frac{1}{4}\right)\pi\hbar$$

At the turning point $E = V(x_t) = mgx_t \rightarrow x_t = E/mg$. Then

$$\begin{aligned}
\int_0^{x_t} p dx &= \int_0^{x_t} \sqrt{2m(E-V)} \\
&= \int_0^{x_t} \sqrt{2m(E-mgx)} dx \\
&= \sqrt{2m^2 g} \int_0^{x_t} \sqrt{(\frac{E}{mg} - x)} dx \\
&= \sqrt{2m^2 g}\left(-\frac{2}{3}\right)\left(\frac{E}{mg} - x\right)^{\frac{3}{2}} |_0^{\frac{E}{mg}} \\
&= \sqrt{2m^2 g}\left(\frac{2}{3}\right)\left(\frac{E}{mg}\right)^{\frac{3}{2}} \\
&= \sqrt{2\frac{m}{g}}\left(\frac{2}{3}\right) E^{\frac{3}{2}} = \left(n - \frac{1}{4}\right)\pi\hbar \\
\rightarrow E &= \left(\frac{mg^2\hbar^2}{2}\right)^{\frac{1}{3}}\left(\frac{3\pi}{2}\right)^{\frac{2}{3}}\left(n - \frac{1}{4}\right)^{\frac{2}{3}}]
\end{aligned}$$

(b) Compare the WKB approximation to the first four energies with the "exact" results that we found in class.

[Let $E_n = \epsilon_n E_0$, where $E_0 = \left(\frac{mg^2\hbar^2}{2}\right)^{\frac{1}{3}}$. Then $\epsilon_n(\text{WKB}) = \left(\frac{3\pi}{2}\right)^{\frac{3}{2}}\left(n - \frac{1}{4}\right)^{\frac{2}{3}}$. $\epsilon_n$(WKB) and the first four zeros of $Ai(z)$ are compared in the table.]

| $n$ | $\epsilon_n$(WKB) | $\epsilon_n$ (Exact) |
|---|---|---|
| 1 | 2.319 | 2.338 |
| 2 | 4.081 | 4.088 |
| 3 | 5.518 | 5.521 |
| 4 | 6.785 | 6.787 |

(c) About how large would the quantum number $n$ have to be to give the ball an average height of, say, 1 meter above the ground? [

According to the Virial Theorem the expectation values of kinetic and potential energy in a stationary state are related:

$$2\langle T\rangle = \langle x\frac{dV}{dx}\rangle$$

For the gravitational potential $x(dV/dx) = V$. Therefore since $E = T + V$ we have that

$$E_n = \frac{3}{2}\langle V\rangle = \frac{3}{2}mg\langle x\rangle$$

The energy of the ball is

$$\begin{aligned} E_n &= \left(\frac{mg^2\hbar^2}{2}\right)^{\frac{1}{3}}\left(\frac{3\pi}{2}\right)^{\frac{2}{3}}\left(n-\frac{1}{4}\right)^{\frac{2}{3}} \\ \rightarrow n &= E_n^{\frac{3}{2}}\left(\frac{2}{mg^2\hbar^2}\right)^{\frac{1}{2}}\left(\frac{2}{3\pi}\right)+\frac{1}{4} \end{aligned}$$

or in terms of $\langle x\rangle$

$$\begin{aligned} n &= \left(\frac{3}{2}mg\langle x\rangle\right)^{\frac{3}{2}}\left(\frac{2}{mg^2\hbar^2}\right)^{\frac{1}{2}}\left(\frac{2}{3\pi}\right)+\frac{1}{4} \\ &= \frac{m}{\hbar}\langle x\rangle^{\frac{3}{2}}(3g)^{\frac{1}{2}}\left(\frac{1}{\pi}\right)+\frac{1}{4} \\ &= \frac{(0.1\text{ kg})}{(1.05\times10^{-34}\text{ J}-\text{s})}(1\text{ m})^{\frac{3}{2}}(3(9.8\text{ m/s}^2))^{\frac{1}{2}}\left(\frac{1}{\pi}\right)+\frac{1}{4} \\ &= 1.6\times10^{34} \end{aligned}$$

4. **Griffiths 8.8**. This problem involves the harmonic oscillator. The potential $V(x)$ and allowed energies are known to be

$$\begin{aligned} V(x) &= 1/2\ m\omega^2x^2, \\ V'(x) &= m\omega^2 x, \\ E &= (n+1/2)\hbar\omega. \end{aligned}$$

The first part requires that we find a turning point $x_2$, such that $V(x_2) = E$. It follows from the equations above that that

$$x_2 = \sqrt{\frac{2(n+1/2)\hbar}{m\omega}}.$$

Next we need to calculate the potential to linear order, we have

$$V_{lin}(x) = (n + 1/2)\hbar\omega + m\omega\sqrt{\frac{2(n+1/2)\hbar}{m\omega}}x.$$

And we need to calculate $d$ such that

$$\begin{aligned}
\frac{V(x_2+d) - V_{lin}(x_2+d)}{V(x_2)} &= 0.01, \\
\frac{x_2^2 + d^2 + 2x_2 d - x_2^2 - 2x_2 d}{x_2^2} &= 0.01, \\
d &= 0.1\ x_2.
\end{aligned}$$

We have that $d = 0.1\ x_2$, and we need to find the smallest $n$ such that $\alpha d \geq 5$. We have

$$\begin{aligned}
\alpha &= \left[\frac{2m}{\hbar^2}V'(x_2)\right]^{\frac{1}{3}}, \\
&= \left[\frac{2m}{\hbar^2}m\omega^2 x_2\right]^{\frac{1}{3}}, \\
\alpha d &= \left[\frac{2m}{\hbar^2}m\omega^2 x_2^4\right]^{\frac{1}{3}} 0.1, \\
&= \left[\frac{2m}{\hbar^2}m\omega^2\frac{4(n+1/2)^2\hbar^2}{m^2\omega^2}\right]^{\frac{1}{3}} 0.1, \\
&= 2(n+1/2)^{\frac{2}{3}}\frac{1}{10} \geq 5, \\
(n+1/2) &\geq (25)^{\frac{3}{2}}, \\
n &\geq (25)^{\frac{3}{2}} - 1/2 \approx 125.
\end{aligned}$$

5. **Griffiths 8.15** .

   (a) We begin by writing the WKB wave functions for each of the regions

$$\psi_i = \frac{1}{\sqrt{|p|}}\left(Ce^{\frac{1}{\hbar}\int_{x_2}^{x}|p(x')|dx'} + De^{-\frac{1}{\hbar}\int_{x_2}^{x}|p(x')|dx'}\right) \quad (2)$$

$$\psi_{ii} = \frac{1}{\sqrt{p}}\left(Ae^{i\frac{1}{\hbar}\int_{x_2}^{x}p(x')dx'} + Be^{-i\frac{1}{\hbar}\int_{x_2}^{x}p(x')dx'}\right) \quad (3)$$

$$\psi'_{ii} = \frac{1}{\sqrt{p}}\left(A'e^{i\frac{1}{\hbar}\int_{x_1}^{x}p(x')dx'} + B'e^{-i\frac{1}{\hbar}\int_{x_1}^{x}p(x')dx'}\right) \quad (4)$$

$$\psi_{iii} = \frac{1}{\sqrt{|p|}}\left(C'e^{\frac{1}{\hbar}\int_{x_1}^{x}|p(x')|dx'} + D'e^{-\frac{1}{\hbar}\int_{x_1}^{x}|p(x')|dx'}\right) \quad (5)$$

Note that we always integrate away from the turning points. Then there are two alternative versions of $\psi_{ii}$, one referenced to $x_1$ and one referenced to $x_2$. We will use $\psi_{ii}$ when we are connecting to $\psi_i$ at $x_2$, and we use $\psi'_{ii}$ when we connect to $\psi_{iii}$ at $x_1$.

In order that $\psi_i$ be finite for large positive $x$, $C = 0$. Then use the connection formulae for a barrier to the right (page 4 of WKB notes) and we find that

$$A = De^{i\pi/4}, \text{ and } B = De^{-i\pi/4} \quad (6)$$

$$\begin{aligned}
\psi_{ii} &= \frac{D}{\sqrt{p}}\left(e^{i\frac{1}{\hbar}\int_{x_2}^{x}p(x')dx' + i\pi/4} + e^{-i\frac{1}{\hbar}\int_{x_2}^{x}p(x')dx' - i\pi/4}\right) \\
&= \frac{2D}{\sqrt{p}}\cos\left(\frac{1}{\hbar}\int_{x_2}^{x}p(x')dx' + \pi/4\right) \\
&= \frac{-2D}{\sqrt{p}}\sin\left(\frac{1}{\hbar}\int_{x_2}^{x}p(x')dx' - \pi/4\right) \\
&= \frac{2D}{\sqrt{p}}\sin\left(\frac{1}{\hbar}\int_{x}^{x_2}p(x')dx' + \pi/4\right)
\end{aligned}$$

Now we can use the connection formulae for a boundary to the left (page 5 of the WKB notes) to relate $A'$ and $B'$ to $C'$ and $D'$. We get that

$$D' = A'e^{-i\pi/4} + B'e^{i\pi/4} \quad (7)$$

$$C' = \frac{1}{2}(A'e^{i\pi/4} + B'e^{-i\pi/4}) \quad (8)$$

The next thing to do is to relate $A'$ and $B'$ to $A$ and $B$. We can rewrite Equation 4 as

$$\psi'_{ii} = \frac{1}{\sqrt{p}}\left(A'e^{i\frac{1}{\hbar}\left(\int_{x_1}^{x_2}p(x')dx' + \int_{x_2}^{x}p(x')dx'\right)} + B'e^{-i\frac{1}{\hbar}\left(\int_{x_1}^{x_2}p(x')dx' + \int_{x_2}^{x}p(x')dx'\right)}\right)$$

Comparison with Equation 3 gives

$$
\begin{aligned}
A' e^{i\theta} &= A && (9)\\
B' e^{-i\theta} &= B && (10)
\end{aligned}
$$

where $\theta = \int_{x_1}^{x_2} p(x')dx'$. Substitution of Equations 6 9 and 10 into Equations 7 and 8 gives

$$
\begin{aligned}
D' &= Ae^{-i\theta}e^{-i\pi/4} + Be^{i\theta}e^{i\pi/4}\\
&= De^{i\pi/4}e^{-i\theta}e^{-i\pi/4} + De^{-i\pi/4}e^{i\theta}e^{i\pi/4}\\
&= 2D\cos\theta\\
C' &= \frac{1}{2}(Ae^{-i\theta}e^{i\pi/4} + Be^{i\theta}e^{-i\pi/4})\\
&= \frac{1}{2}D(e^{i\pi/4}e^{-i\theta}e^{i\pi/4} + e^{-i\pi/4}e^{i\theta}e^{-i\pi/4})\\
C' &= D\sin(\theta)
\end{aligned}
$$

Finally

$$
\begin{aligned}
\psi_{iii} &= \frac{D}{\sqrt{|p|}}\left(\sin\theta e^{\frac{1}{\hbar}\int_{x_1}^{x}|p(x')|dx'} + 2\cos\theta e^{-\frac{1}{\hbar}\int_{x_1}^{x}|p(x')|dx'}\right)\\
&= \frac{D}{\sqrt{|p|}}\left(\sin\theta e^{-\frac{1}{\hbar}\int_{x}^{x_1}|p(x')|dx'} + 2\cos\theta e^{\frac{1}{\hbar}\int_{x}^{x_1}|p(x')|dx'}\right) \quad (11)
\end{aligned}
$$

(b) If the wave function is antisymmetric then $\psi_{iii}(0) = 0$ and we see from Equation 11 that

$$
\begin{aligned}
\psi_{iii}(0) &= \frac{D}{\sqrt{|p|}}\left(\sin\theta e^{-\frac{1}{\hbar}\int_{0}^{x_1}|p(x')|dx'} + 2\cos\theta e^{\frac{1}{\hbar}\int_{0}^{x_1}|p(x')|dx'}\right)\\
\psi_{iii}(0) &= 0 \rightarrow \tan\theta = -2e^{\frac{2}{\hbar}\int_{0}^{x_1}|p(x')|dx'} = -2e^{\frac{1}{\hbar}\int_{-x_1}^{x_1}|p(x')|dx'} = -2e^{\phi}
\end{aligned}
$$

If $\psi_{iii}(x)$ is symmetric then

$$
\begin{aligned}
\psi_{iii}(x_1) &= \psi_{iii}(-x_1)\\
\frac{D}{\sqrt{|p|}}(\sin\theta + 2\cos\theta) &= \frac{D}{\sqrt{|p|}}\left(\sin\theta e^{-\frac{1}{\hbar}\int_{-x_1}^{x_1}|p(x')|dx'} + 2\cos\theta e^{\frac{1}{\hbar}\int_{-x_1}^{x_1}|p(x')|dx'}\right)\\
\sin\theta + 2\cos\theta &= \left(\sin\theta e^{-\phi} + 2\cos\theta e^{\phi}\right)
\end{aligned}
$$

$$
\begin{aligned}
\rightarrow \tan\theta &= 2\frac{e^{\phi}-1}{1-e^{-\phi}} \\
&= 2\frac{e^{\phi/2}(e^{\phi/2}-e^{-\phi/2})}{e^{-\phi/2}(e^{\phi/2}-e^{-\phi/2})} \\
&= 2e^{\phi}
\end{aligned}
$$

(c) The quantization condition is

$$\tan\theta = \pm 2e^{\phi}$$

If $\theta = (n+\frac{1}{2})\pi + \epsilon$ where $\epsilon \ll 1$ then

$$
\begin{aligned}
\tan\left((n+\frac{1}{2})\pi+\epsilon\right) \sim -\frac{1}{\epsilon} \\
&= \pm 2e^{\phi} \\
\rightarrow \epsilon &= \mp\frac{1}{2}e^{-\phi} \\
\rightarrow \theta &= (n+\frac{1}{2})\pi + \epsilon = (n+\frac{1}{2})\pi \mp \frac{1}{2}e^{-\phi}
\end{aligned}
$$

(d) We have that

$$
\begin{aligned}
\theta &= \frac{1}{\hbar}\int_{x_1}^{x_2} p(x')dx' \\
&= \frac{1}{\hbar}\int_{x_1}^{x_1} \sqrt{2m(E-V(x')}dx' \\
&= \frac{1}{\hbar}\int_{x_1}^{x_2} \sqrt{2m(E-\frac{1}{2}m\omega^2(x-a)^2)}dx' \\
&= \frac{1}{\hbar}m\omega\int_{x_1}^{x_2} \sqrt{y_0^2-y^2}dy
\end{aligned}
$$

where $y = x-a$, and $y_0^2 = \frac{2E}{m\omega^2}$. $x_1$ and $x_2$ are the turning points defined by

$$E = \frac{1}{2}m\omega^2(x_t-a)^2$$

or $y_t = \pm y_0$. So

$$\begin{aligned}
\theta &= \frac{1}{\hbar} m\omega \int_{-y_0}^{y_0} \sqrt{y_0^2 - y^2} dy \qquad (12)\\
&= \frac{m\omega}{\hbar}\frac{1}{2}\left(\sqrt{y_0^2-y^2}y + y_0^2 \tan^{-1}\left(\frac{y}{\sqrt{y_0^2-y^2}}\right)\right)_{-y_0}^{y_0}\\
&= \frac{m\omega}{\hbar 2}(y_0^2\pi)\\
&= \frac{m\omega}{\hbar 2}\frac{2E\pi}{m\omega^2} = \frac{E\pi}{\hbar\omega}
\end{aligned}$$

The

$$\begin{aligned}
\frac{E_n^{\pm}\pi}{\hbar\omega} &= (n+\frac{1}{2})\pi \mp \frac{1}{2}e^{-\phi}\\
\to E_n^{\pm} &= (n+\frac{1}{2})\hbar\omega \mp \frac{\hbar\omega}{2\pi}e^{-\phi}
\end{aligned}$$

(e)

$$\begin{aligned}
\Psi(x,t) &= \frac{1}{\sqrt{2}}(\psi_n^+ e^{-iE_n^+ t/\hbar} + \psi_n^- e^{-E_n^- t/\hbar})\\
|\Psi(x,t)|^2 &= \frac{1}{2}(|\psi_n^+|^2 + |\psi_n^-|^2 + 2\psi_n^+\psi_n^- \cos(E_n^+ - E_n^-)t/\hbar)\\
&= \frac{1}{2}(|\psi_n^+|^2 + |\psi_n^-|^2 + 2\psi_n^+\psi_n^- \cos 2\pi t/\tau)
\end{aligned}$$

where

$$\tau = \frac{2\pi\hbar}{(E_n^+ - E_n^-)} = \frac{2\pi\hbar}{\frac{\hbar\omega e^{-\phi}}{\pi}} = \frac{2\pi^2}{\omega}e^{\phi}$$

(f) We have an integral like the one in Equation 12, but now we set the limits to be $x = 0 \to y_{max} = -a$, and $x_1 = y_0$. Also, in this range, $y$ is always greater than $y_0$ so we have

$$\begin{aligned}
\phi &= \frac{2}{\hbar}\int_{-x_1}^{0} |p(x')|dx' = \frac{2m\omega}{\hbar}\int_{-y_0}^{y_{max}} \sqrt{y^2 - y_0^2}dy\\
&= \frac{m\omega}{\hbar}\left(y\sqrt{y^2-y_0^2} - y_0^2\log\left(y + \sqrt{y^2-y_0^2}\right)\right)_{-y_0}^{y_{max}}\\
&= \frac{m\omega}{\hbar}\left[\left(a\sqrt{a^2-y_0^2} - y_0^2\log\left(a+\sqrt{a^2-y_0^2}\right)\right) - y_0^2\log(y_0)\right]
\end{aligned}$$

Now since $y_0^2 = \frac{2E}{m\omega^2}$ and $a^2 = \frac{2V(0)}{m\omega^2}$, the limit where $V(0) \gg E$, is equivalent to $a^2 \gg y_0^2$ and we get that

$$\phi = \frac{m\omega}{\hbar}\frac{2V(0)}{m\omega^2} = \frac{m\omega a^2}{\hbar}$$

6. **Symmetry/Antisymmetry of WKB wavefunctions** In this problem you need to show explicitly that for the WKB wavefunctions, we have that $\psi(-x) = \pm\psi(x)$. In general, this could be a tedious task, were it not for the connection formulas derived in class. In the lecture notes, we found the quantization condition $\cos\phi = 0 \to \phi = (n+1/2)\pi$. (Refer to Eq. (18) and Eq. (19)). The other condition from Eq. (18) is:

$$\frac{D}{C} = \sin\phi.$$

Using the quantization condition for $\cos\phi$, we see that $\sin\phi = \pm 1$ (and this is true in general where we have not yet specified that the potential is symmetric). The key observation at this stage is that $C/D = \pm 1$. The next step is to calculate $\psi(x)$ and $\psi(-x)$. We calculate $\psi(x)$ from the right barrier, and find that $A = D\exp(i\pi/4)$ and $B = D\exp(-i\pi/4)$, and because we are integrating from right to left, the limits of the integration go from $x_2$ to $x$. To calculate $\psi(-x)$ we calculate it from the left barrier, where we know that $A' = C\exp(-i\pi/4)$ and $B' = C\exp(i\pi/4)$. And in this case the integration goes from $x_1$ to $x$. Now we impose the symmetry of the potential. Since $V(x) = V(-x)$, we have that $p(x) = p(-x)$, and that

$$\int_{x_1}^{x} = \int_{x}^{x_2} = -\int_{x_2}^{x}.$$

This reversal of the sign of the integral means that we have to compare the co-efficients of $A'$ and $B$, and similarly the co-efficients $B'$ and $A$, where the prime refers to the co-efficients of $\psi(-x)$ and the non-primed refers to $\psi(x)$. Or to write this out explicitly, we have

$$\begin{aligned}
\psi(x) &= A\exp\left(\frac{i}{\hbar}\int_{x_2}^{x} p(x')dx'\right) + B\exp\left(-\frac{i}{\hbar}\int_{x_2}^{x} p(x')dx'\right) \\
\psi(-x) &= A'\exp\left(\frac{i}{\hbar}\int_{x_1}^{x} p(x')dx'\right) + B'\exp\left(-\frac{i}{\hbar}\int_{x_1}^{x} p(x')dx'\right) \\
&= A'\exp\left(-\frac{i}{\hbar}\int_{x_2}^{x} p(x')dx'\right) + B'\exp\left(\frac{i}{\hbar}\int_{x_2}^{x} p(x')dx'\right)
\end{aligned}$$

$$A = D\exp\left(\frac{i\pi}{4}\right) \qquad B = D\exp\left(\frac{-i\pi}{4}\right)$$
$$B' = C\exp\left(\frac{i\pi}{4}\right) \qquad A' = C\exp\left(\frac{-i\pi}{4}\right)$$

We see that

$$\frac{\psi(x)}{\psi(-x)} = \frac{C}{D} = \pm 1.$$

This proves that for a symmetric potential, the WKB wavefunctions are either even or odd.