1. **Angular momentum** 1

Consider a system with total angular momentum 1 and with basis vectors
\[ \chi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \chi_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \chi_{-1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \]

(a) Derive matrix representations of \( L_\pm, L_z, L^2 \) and \( L_y \).

(b) Any state \( \psi \) with angular momentum \( l = 1 \) can be written as a linear combination of the base states.
\[ \psi = a\chi_1 + b\chi_0 + c\chi_{-1} \]
where \( a, b, \) and \( c \) are the amplitudes for the various components of angular momentum with respect to the z-axis. In a new coordinate system that is related to the original by a rotation by an angle \( \theta \) about the y-axis, we can write
\[ \psi = a'\chi'_1 + b'\chi'_0 + c'\chi'_{-1} \]

\( a', b', \) and \( c' \) are amplitudes for components of angular momentum along the \( z' \) axis. The angle between the \( z \) axis and the \( z' \) axis is \( \theta \). Derive the rotation matrix \( R_y(\theta) \) that relates \( a, b \) and \( c \) with \( a', b', \) and \( c' \).

2. **Griffiths 4.19.**

(a) Starting with the canonical commutation relations for position and momentum
\[ [r_i, p_j] = i\hbar \delta_{ij}, \quad \text{where } r_1 = x, r_2 = y, r_3 = z, \text{ and } p_1 = p_x, p_2 = p_y, p_3 = p_z \]
work out the following commutators:
\[ [L_z, x] = i\hbar y, \quad [L_z, y] = -i\hbar x, \quad [L_z, z] = 0, \]
\[ [L_z, p_x] = i\hbar p_y, \quad [L_z, p_y] = -i\hbar p_x, \quad [L_z, p_z] = 0. \quad (1) \]

(b) Use these results to obtain \([L_z, L_z] = i\hbar L_y \) directly from
\[ L_x = yp_z - zp_y, \quad L_y = zp_x - xp_z, \quad L_z = xp_y - yp_x. \]
(c) Evaluate the commutators \([L_z, r^2]\) and \([L_z, p^2]\) (where, of course, 
\(r^2 = x^2 + y^2 + z^2\) and \(p^2 = p_x^2 + p_y^2 + p_z^2\)).

(d) Show that the Hamiltonian \(H = (p^2/2m) + V\) commutes with all
three components of \(L\), provided that \(V\) depends only on \(r\). (Thus
\(H, L^2,\) and \(L_z\) are mutually compatible observables.)


(a) Prove that for a particle in a potential \(V(r)\) the rate of change of
the expectation value of the orbital angular momentum \(L\) is equal
to the expectation value of the torque:
\[
\frac{d}{dt} \langle L \rangle = \langle N \rangle,
\]
where
\[
N = r \times (-\nabla V).
\]
(This is the rotational analog of Ehrenfest’s theorem.)

(b) Show that \(d\langle L\rangle/dt = 0\) for any spherically symmetric potential.
(This is one form of the quantum statement of conservation of angular momentum.)

4. Griffiths 4.22

(a) What is \(L_\pm Y_l^l\)? (No calculation allowed!)

(b) Use the result of (a), together with the fact that
\[
L_\pm = \pm \hbar e^{\pm i\phi} \left( \frac{\partial}{\partial \theta} \pm \cot \theta \frac{\partial}{\partial \phi} \right),
\]
and the fact that \(L_z Y_l^l = \hbar l Y_l^l\), to determine \(Y_l^l(\theta, \phi)\), up to a
normalization constant.

(c) Determine the normalization constant by direct integration. Compare your answer for \(l = 3\) to what appears in the table on page 139.

5. Griffiths 4.27. An electron is in the spin state
\[
\chi = A \left( \frac{3i}{4} \right).
\]
(a) Determine the normalization constant $A$.

(b) Find the expectation values of $S_x$, $S_y$, and $S_z$.

(c) Find the "uncertainties" $\sigma_{S_x}$, $\sigma_{S_y}$, and $\sigma_{S_z}$. (Note: These sigmas are standard deviations, not Pauli matrices!)

(d) Confirm that your results are consistent with all three uncertainty principles, namely

$$\sigma_{S_x} \sigma_{S_y} \geq \frac{\hbar}{2} |\langle L_z \rangle|$$

and its cyclic permutations.

6. Griffiths 4.28. For the most general normalized spinor $\chi$ where

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\chi_+ + b\chi_-,$$

with

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ and } \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

compute $\langle S_x \rangle$, $\langle S_y \rangle$, $\langle S_z \rangle$, $\langle S_x^2 \rangle$, $\langle S_y^2 \rangle$, and $\langle S_z^2 \rangle$. Check that $\langle S_x^2 \rangle + \langle S_y^2 \rangle + \langle S_z^2 \rangle = \langle S^2 \rangle$.