
(a) Find \( \langle r \rangle \) and \( \langle r^2 \rangle \) for an electron in the ground state of hydrogen. Express your answers in terms of the Bohr radius.

The ground state of the Hydrogen wavefunction can be written as

\[
\psi_{100} = \frac{\exp\left(- \frac{r}{a}\right)}{\sqrt{\pi a^3}}
\]

where \( a \) is the Bohr radius. We can then calculate

\[
\langle r \rangle = \frac{1}{\pi a^3} \int r^3 e^{-2r/a} dr d\Omega = 4a \int_0^\infty u^3 e^{-2u} du = \frac{3a}{2}.
\]

Similarly,

\[
\langle r^2 \rangle = 4a^2 \int u^4 e^{-2u} du = 3a^2.
\]

(b) Find \( \langle x \rangle \) and \( \langle x^2 \rangle \) for an electron in the ground state of hydrogen.

*Hint*: This requires no new integration - note that \( r^2 = x^2 + y^2 + z^2 \), and exploit the symmetry of the ground state.

We have that

\[
\langle x \rangle = \int_0^{2\pi} d\phi \int_0^\pi d(\cos \theta) \int_0^\infty \frac{e^{-2r/a}}{\sqrt{\pi a^3}} r^2 dr = 0
\]

and by symmetry

\[
\langle x^2 \rangle = \langle y^2 \rangle = \langle z^2 \rangle = \langle r^2 \rangle / 3 = a^2.
\]

(c) Find \( \langle x^2 \rangle \) in the state \( n = 2, l = 1, m = 1 \). *Warning*: This is not symmetrical in \( x, y, z \). Use \( x = r \sin \theta \cos \phi \).

For part (c), we write

\[
\psi_{211} = -\sqrt{\frac{3}{8\pi}} \frac{1}{\sqrt{24a^5}} \frac{r}{a} e^{-\frac{r}{a}} \sin \theta e^{i\phi}.
\]

To calculate the expectation value

\[
\langle x^2 \rangle = \frac{3}{8\pi} \left( \frac{1}{24a^5} \right) \int_0^{2\pi} \left( \frac{r}{a} \right)^2 e^{-\frac{r}{a}} \sin^2 \theta (r^2 \sin^2 \theta \cos^2 \phi) (r^2 \sin \theta dr d\theta d\phi)
\]

\[
= \frac{3}{8\pi} \left( \frac{1}{24a^5} \right) \int_0^{2\pi} \cos^2 \phi d\phi \int_0^\pi \sin^5 \theta d\theta \int_0^{\infty} r^6 e^{-\frac{r}{a}} dr
\]
\[
\begin{align*}
&= \frac{3}{8\pi} \frac{1}{24a^5} (\pi) \int_{-1}^{1} (1 - x^2)^2 dx \int_{0}^{\infty} a^7 u^6 e^{-u} du \\
&= \frac{3}{8\pi} \frac{1}{24a^5} (\pi) \left( \frac{16}{15} \right) \int_{0}^{\infty} a^7 u^6 e^{-u} du \\
&= \frac{3}{8\pi} \frac{1}{24a^5} (\pi) \left( \frac{16}{15} \right) a^7 6! \\
&= 12a^2
\end{align*}
\]

2. Griffiths 4.16. In this problem notice that

\[ V(r) = \frac{-Ze^2}{4\pi\epsilon_0} \frac{1}{r} \]

is just the same potential as the hydrogen atom with \( e^2 \rightarrow Ze^2 \). Which means that we can use all the results of the Hydrogen atom making this substitution. Looking at the dependence of these functions on \( e^2 \), we can write down the answers as:

\[ E_n(Z) = Z^2 \epsilon_n; a(z) = \frac{a}{Z}; R(Z) = Z^2 R \]

\[ \frac{1}{\lambda} \bigg|_{\text{Lyman}} = \left( \frac{4}{3R}, \frac{1}{R} \right) \rightarrow \left( \frac{4}{3Z^2 R}, \frac{1}{Z^2 R} \right) \]

For \( Z = 2 \), \((2.28\times10^{-8}m, 3.04\times10^{-8}m) \in \text{ultraviolet}\)

For \( Z = 3 \), \((1.01\times10^{-8}m, 1.35\times10^{-8}m) \in \text{ultraviolet}\)


(a) Find the eigenvalues and eigenspinors of \( S_y \).

[The eigenvalues of \( S_y \) are \( \pm \hbar/2 \). The eigenvalues of a spin \( \frac{1}{2} \) matrix are \( \pm \frac{1}{2} \) regardless of axis. Then]

\[ \frac{\hbar}{2} \sigma_y \chi_{\pm} = \pm \frac{\hbar}{2} \chi_{\pm} \]

\[ \rightarrow \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ b \end{pmatrix} = \pm \begin{pmatrix} 1 \\ b \end{pmatrix} \]

\[ \rightarrow b = \pm i \]

The normalized eigenvectors are

\[ \chi^{(y)}_{\pm} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \end{pmatrix} \]
(b) If you measured $S_y$ on a particle in the general state $\chi$

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\chi_+ + b\chi_-,$$

what values might you get, and what is the probability of each? Check that the probabilities add up to 1. Note: $a$ and $b$ need not be real.

[We can write that]

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\chi_+ + b\chi_-$$

or in the $y$-basis

$$\chi = \begin{pmatrix} a' \\ b' \end{pmatrix} = a'\chi_+^{(y)} + b'\chi_-^{(y)}$$

The probability that we find the particle with spin $+\frac{\hbar}{2}$, that is, $P_{\frac{1}{2}}$ is

$$P_{\frac{1}{2}} = |a\chi_+^{(y)} + b\chi_-^{(y)}|^2$$

$$= \left| a\begin{pmatrix} 1 \\ -i \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right|^2$$

$$= \left| \frac{a}{\sqrt{2}} + \frac{ib}{\sqrt{2}} \right|^2$$

$$= \frac{1}{2}(|a|^2 + i(a^*b - ib^*a) + |b|^2)$$

The probability that we find the particle with spin $-\frac{\hbar}{2}$ is

$$P_{-\frac{1}{2}} = \left| a\chi_+^{(y)} + b\chi_-^{(y)} \right|^2$$

$$= \left| a\begin{pmatrix} 1 \\ i \end{pmatrix} + b\begin{pmatrix} 0 \\ -i \end{pmatrix} \right|^2$$

$$= \left| \frac{a}{\sqrt{2}} + \frac{-ib}{\sqrt{2}} \right|^2$$

$$= \frac{1}{2}( |a|^2 - i(a^*b + ib^*a) + |b|^2)$$

$$P_{\frac{1}{2}} + P_{-\frac{1}{2}} = |a|^2 + |b|^2 = 1.$$
(c) If you measured $S_y^2$, what values might you get, and with what probabilities?

$[S_y^2 = \frac{\hbar}{4}$ for either of the two eigenstates. So we measure $(S_y)^2 = \frac{\hbar}{4}$ with unit probability.]

4. **Griffiths 4.30.** We can begin by constructing

$$S_r = \frac{\hbar}{2} [\sin \theta \cos \phi \sigma_x + \sin \theta \sin \phi \sigma_y + \cos \theta \sigma_z],$$

$$= \frac{\hbar}{2} \begin{pmatrix} \cos \theta & \sin \theta e^{-i\phi} \\ \sin \theta e^{i\phi} & -\cos \theta \end{pmatrix}.$$

Solving for the eigenvalues, we have that $(\cos \theta - \lambda)(\cos \theta + \lambda) + \sin^2 \theta = 0$, giving us eigenvalues $\lambda = \pm \hbar/2$. The eigenvectors are found using the normal procedure. For $\chi_+ = (x,y)$, we have that $(\cos \theta - 1)x + \sin \theta \exp(-i\phi)y = 0$, or after applying a trig identity, $y \cos(\theta/2) = \exp(i\phi) \sin(\theta/2)x$. And normalization requires that $|x|^2 + |y|^2 = 1$, or $|x|^2(1 + \tan^2(\theta/2)) = 1$, giving $x = \cos(\theta/2)$ and $y = \sin(\theta/2) \exp(i\phi)$. And similarly for $\chi_-$. The answers are:

$$\chi_+ = \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi} \sin(\theta/2) \end{pmatrix}, \quad \chi_- = \begin{pmatrix} \sin(\theta/2) \\ -e^{i\phi} \cos(\theta/2) \end{pmatrix}.$$

5. **Griffiths 4.33.** An electron is at rest in an oscillating magnetic field

$$B = B_0 \cos(\omega t) \hat{k},$$

where $B_0$ and $\omega$ are constants.

(a) Construct the Hamiltonian matrix for this system.

[The Hamiltonian for this system can be written as $H = -\mu \cdot B = -\gamma B \cdot S = -\gamma B \cdot S = -\gamma B_0 \cos(\omega t) \frac{\hbar}{2} \sigma_z$.]

(b) The electron starts out (at $t = 0$) in the spin-up state with respect to the $x$-axis (that is: $\chi(0) = \chi^{(x)}_+$. Determine $\chi(t)$ at any subsequent time. *Beware*: This is a time-dependent Hamiltonian, so you cannot get $\chi(t)$ in the usual way from stationary states. Fortunately, in this case you can solve the time-dependent Schrodinger equation

$$i\hbar \frac{\partial \chi}{\partial t} = H \chi,$$
directly.

[From Schrodinger’s equation we get that

\[
i\hbar \frac{\partial}{\partial t} \begin{pmatrix} a \\ b \end{pmatrix} = -\gamma B_0 \cos(\omega t) \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}
\]

where \( \chi(t) = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} \). Then we have a pair of differential equations

\[
i\hbar \frac{\partial a}{\partial t} = -\gamma B_0 \cos(\omega t) \frac{\hbar}{2} a
\]

\[
\Rightarrow \frac{da}{a} = \frac{i\gamma B_0 \cos(\omega t)}{2} \frac{\hbar}{2} dt
\]

\[
\Rightarrow a = a(0) \exp\left( \frac{i\gamma B_0 \sin(\omega t)}{2\omega} \right)
\]

Similarly

\[
i\hbar \frac{\partial b}{\partial t} = \gamma B_0 \cos(\omega t) \frac{\hbar}{2} b
\]

\[
\Rightarrow b = b(0) \exp\left( \frac{i\gamma B_0 \sin(\omega t)}{2\omega} \right)
\]

At \( t = 0 \),

\[
\chi(0) = \begin{pmatrix} a(0) \\ b(0) \end{pmatrix} = \chi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]

Therefore

\[
\chi(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} \exp\left( \frac{-i\gamma B_0 \sin(\omega t)}{2\omega} \right) \\ \exp\left( \frac{i\gamma B_0 \sin(\omega t)}{2\omega} \right) \end{pmatrix}
\]

(c) Find the probability of getting \( \hbar /2 \), if you measure \( S_x \).

[The probability to get \( S_x = -\hbar /2 \) is given by the projection of \( \chi(t) \) onto the eigenstate of \( S_z \) with eigenvalue \( -\hbar /2 \), namely \( \chi^{(x)} = \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \). Then the probability is

\[
|\langle \chi^{(x)} | \chi(t) \rangle|^2 = \left| \frac{1}{2} \left( e^{-\frac{i\gamma B_0 \sin(\omega t)}{2\omega}} - e^{\frac{i\gamma B_0 \sin(\omega t)}{2\omega}} \right) \right|^2
\]

\[
= \sin^2 \frac{\gamma B_0 \sin(\omega t)}{2\omega}
\]
(d) What is the minimum field \( (B_0) \) required to force a complete flip in \( S_x \)?

[We see that the minimum field for a spin flip is that required so that \( |\langle \chi^-(x) | \chi(t) \rangle|^2 = 1 \) which will occur only if \( \frac{\gamma B_0}{2 \omega} \geq \frac{\pi}{2} \), or if \( B_0 \geq \frac{\gamma \pi}{\omega} \).

6. Griffiths 4.36. This problem involves reading out values from the Clebsh-Gorden Table on pg. 168 of Griffiths. Part (a) asks that we find the co-efficients of the following product in which the total spin is 3 and the z-component is 1.

\[
|3, 1\rangle = (?)|1, 1\rangle \otimes |2, 0\rangle + (?)|1, 0\rangle \otimes |2, 1\rangle + (?)|1, -1\rangle \otimes |2, 2\rangle.
\]

Looking at the table we can fill in the co-efficients as

\[
|3, 1\rangle = \sqrt{\frac{6}{15}}|1, 1\rangle \otimes |2, 0\rangle + \sqrt{\frac{8}{15}}|1, 0\rangle \otimes |2, 1\rangle + \sqrt{\frac{1}{15}}|1, -1\rangle \otimes |2, 2\rangle.
\]

We can then read off the probabilities of the z-component of the spin-2 particle as \( P(-2\hbar) = 0, P(-1\hbar) = 0, P(0) = 6/15, P(1\hbar) = 8/15, P(2\hbar) = 1/15. \)

For part (b), we have to add the angular momentum for an electron with orbital ket \( |1, 0\rangle \) and spin ket \( |1/2, -1/2\rangle \). Again this is just looking up the Clebsh-Gordon table to find that

\[
|1, 0\rangle \otimes |1/2, -1/2\rangle = \sqrt{\frac{2}{3}}|3/2, -1/2\rangle + \sqrt{\frac{1}{3}}|1/2, -1/2\rangle.
\]

We have that with probability 2/3 we will have \( J = 3/2 \) or \( J^2 = J(J + 1) = 15/4 \hbar^2 \), and probability 1/3 that we will have \( J = 1/2 \), or \( J^2 = J(J + 1) = 3/4 \hbar^2 \).

7. Show that

\[
e^{i(\sigma \cdot \hat{n})\alpha/2} = \cos(\alpha/2) + i(\hat{n} \cdot \sigma) \sin(\alpha/2)
\]

where the unit vector

\[
\hat{n} = \sin \theta \cos \phi \hat{i} + \sin \theta \sin \phi \hat{j} + \cos \theta \hat{k}
\]
The operator \( \exp(i(\sigma \cdot \hat{n})\alpha/2) \) effects a rotation of the spinor \( \chi \) through the angle \( \alpha \) about the axis \( \hat{n} \).

[We start with

\[
\exp(i\hat{n} \cdot \sigma\alpha/2) = I + i\hat{n} \cdot \sigma\alpha/2 + \frac{1}{2} \left( i\hat{n} \cdot \sigma\alpha/2 \right)^2 + \frac{1}{3!} \left( i\hat{n} \cdot \sigma\alpha/2 \right)^3 + \ldots
\]

Now

\[
(\hat{n} \cdot \sigma)^2 = (n_x\sigma_x + n_y\sigma_y + n_z\sigma_z)^2
\]

\[
= (n_x^2 + n_y^2 + n_z^2)I
\]

\[
= +n_xn_y(\sigma_x\sigma_y + \sigma_y\sigma_x) + n_xn_z(\sigma_x\sigma_z + \sigma_z\sigma_x) + n_zn_y(\sigma_z\sigma_y + \sigma_y\sigma_z)
\]

\[
= I
\]

where \( I \) is the identity matrix. Then we have

\[
\exp(i\hat{n} \cdot \sigma\alpha/2) = I + i\hat{n} \cdot \sigma\alpha/2 - \frac{1}{2} \left( \frac{\alpha}{2} \right)^2 - i\frac{1}{3!} (\hat{n} \cdot \sigma) \left( \frac{\alpha}{2} \right)^3 + \ldots
\]

\[
\exp\left(\frac{i\hat{n} \cdot \sigma\alpha}{2}\right) = I \left( 1 - \frac{(\alpha/2)^2}{2!} + \cdots \right) + i(\hat{n} \cdot \sigma) \left( \frac{\alpha}{2} - \frac{(\alpha/2)^3}{3!} + \cdots \right)
\]

\[
= I \cos\left( \frac{\alpha}{2} \right) + i(\hat{n} \cdot \sigma) \sin\left( \frac{\alpha}{2} \right),
\]