P443 HW #9
Due April 2, 2008

1. **Griffiths 5.35**. Certain cold stars (called **white dwarfs**) are stabilized against gravitational collapse by the degeneracy pressure of their electrons.

$$P = \frac{(3\pi^2)^{2/3}\hbar^2}{5m}\rho^{5/3}$$

Assuming constant density, the radius R of such an object can be calculated as follows:

   (a) Write the total electron energy

$$E_{tot} = \frac{\hbar^2(3\pi^2 Nq)^{5/3}}{10\pi^2 m}V^{-2/3}$$

   in terms of the radius, the number of nucleons (protons and neutrons) N, the number of electrons per nuclon q, and the mass of the electron m.

   (b) Look up, or calculate, the gravitational energy of a unifomly dense sphere. Express your answer in terms of G (the constant of universal gravitation), R, N, and M (the mass of a nucleon). Note that the gravitational energy is *negative*.

   (c) Find the radius for which the total energy, (a) plus (b), is a minimum. *Answer* :

$$R = \left(\frac{9\pi}{4}\right)^{2/3} \frac{\hbar^2 q^{5/3}}{GmM^2N^{1/3}}.$$

   (Note that the radius *decreases* as the total mass *increases*!) Put in the actual numbers, for everything except N, using $q = 1/2$. *Answer*: $R = 7.6 \times 10^{25} N^{-1/3}$ m.

   (d) Determine the radius, in kilometers, of a white dwarf with the mass of the sun.

   (e) Determine the Fermi energy, in electron volts, for the white dwarf in (d), and compare it with the rest energy of an electron.

2. **Griffiths 6.5**. Consider a charged particle in a one-dimensional harmonic oscillator potential. Suppose that we turn on a weak electric field (E), so that the potential energy is shifted by an amount $H' = -qEx$.

   (a) Show that there is no first-order change in the energy levels, and calculate the second-order correction. (*Hint*: You may have to evaluate matrix elements like $\langle n \mid x \mid n' \rangle$).

   (b) The Schrodinger equation can be solved directly in this case, by a change of variables: $x' \equiv x - (qE/m\omega^2)$. Find the exact energies, and show that they are consistent with the perturbation theory approximation.

3. **Griffiths 6.12**. Use the virial theorem, $(2\langle T \rangle = \langle \mathbf{r} \cdot \nabla V \rangle)$, to prove that for the hydrogen atom

$$\langle \frac{1}{r} \rangle = \frac{1}{n^2 a}.$$

4. **Griffiths 6.14**. Find the (lowest-order) relativistic correction to the energy levels of the one-dimensional harmonic oscillator.

5. **Griffiths 6.25**. Work out the matrix elements of H'_Z and H'_{fs}, and construct the W-matrix given in the text (page 281), for $n = 2$.

6. **Griffiths 6.33**. Suppose the Hamiltonian H, for a particular quantum system, is a function of some parameter λ; let $E_n(\lambda)$ and $\psi_n(\lambda)$ be the eigenvalues and eigenfunctions of $H(\lambda)$. The Feynman-Hellman theorem states that

$$\frac{\partial E_n}{\partial \lambda} = \left\langle \psi_n | \frac{\partial H}{\partial \lambda} | \psi_n \right\rangle$$

   The effective Hamiltonian for the radial wave functions of hydrogen is

$$H = -\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2} - \frac{e^2}{4\pi\epsilon_0} \frac{1}{r},$$

   and the eigenvalues are

$$E_n = -\frac{me^4}{32\pi^2 \epsilon_0^2 \hbar^2 (j_{max} + l + 1)^2}.$$

(a) Use $\lambda = e$ in the Feynman-Hellmann theorem to obtain $\langle 1/r \rangle$. (Griffiths Equation 6.55)

(b) Use $\lambda = l$ to obtain $\langle 1/r^2 \rangle$. (Griffiths Equation 6.56)

7. **Griffiths 6.36**. When an atom is plced in a uniform external electric field $\mathbf{E}_{ext}$, the energy levels are shifted - a phenomenon know as the **Stark effect**. In this problem we analyze the Stark effect for the $n = 1$ and $n = 2$ states of hydrogen. Let the field point in the z direction, so the potential energy of the electron is

$$H'_S = eE_{ext}z = eE_{ext}r\cos\theta.$$

Treat this as a perturbation on the Bohr Hamiltonian. (Ignore spin and neglect fine structure.)

(a) Show that the ground state energy is not affected by this perturbation, in first order.

(b) The first excited state is 4-fold degenerate: $\psi_{200}, \psi_{211}, \psi_{210}, \psi_{21-1}$. Using degenerate perturbation theory, determine the first-order corrections to the energy. Into how many levels does E_2 split?

(c) What are the "good" wave functions for part (b)? Find the expectation value of the electric dipole moment ($\mathbf{p}_e = -e\mathbf{r}$) in each of these "good" states. Notice that the results are independent of the applied field - evidently hydrogen in its first excited state can carry a *permanent* electric dipole moment.

Hint : Most of the integrals in this problem are zero. Study each one carefully before doing any calculations. *Partial answer* : $W_{13} = W_{31} = -3eaE_{ext}$; all other elements are zero.

8. Positronium is an atom consisting of an electron and a positron. Its energy levels are the same as for hydrogen except for the reduced mass. Consider the hyperfine structure in the ground state ($l = 0$). Then $H_{HFS} = \alpha\vec{s}_1 \cdot \vec{s}_2$ where $\vec{s}_1$ and $\vec{s}_2$ are the spins of electron and positron and $\vec{s} = \vec{s}_1 + \vec{s}_2$. The triplet ($s = 1$) state has higher energy than the single $s = 0$ state. The energy difference is Δ. An external magnetic field is applied. The perturbation due to the external field is

$$H_B = \frac{eB}{m}(s_{1z} - s_{sz})$$

(Since electron and positron have opposite charge, they also have opposite magnetic moments.) Solve for the energy levels for the $(l = 0)$ states when the two perturbations are of the same order. $H_B \sim H_{HFS}$.