Physics 443, Solutions to PS 9

1. Griffiths 5.35. Using the assumption that the volume of the star is $V = 4\pi R^3/3$, we can plug this into the equation for the total energy to get that

$$E_{\text{electron}} = \frac{2\hbar^2}{15\pi m R^2}(\frac{9\pi N q}{4})^{\frac{5}{3}}.$$

We also have that the gravitational energy is given by

$$E_{\text{gravity}} = -\frac{3}{5}\mathcal{G}\frac{(NM)^2}{R}.$$

We can add these two to get the total energy. The condition we are looking for is when $dE/dR = 0$. Plugging in and solving for R we find that

$$R = \left(\frac{9\pi}{4}\right)^{\frac{2}{3}} \frac{\hbar^2 q^{5/3}}{\mathcal{G} m M^2 N^{\frac{1}{3}}}.$$

Substituting for numerical values, we get that $R = 7.58\text{x}10^{25}\text{N}^{-1/3}$. Using that the mass of the sun $M_s = 2\text{x}10^{30}$ Kg, we have that $R = 7.16\text{x}10^6$ meters. For the last part, we know that the Fermi energy is given by $(\hbar^2/2m)(9\pi^2 N q/(4\pi R^3))^{2/3}$. This is about $0.194Mev$, which is approaching the rest mass of the electron $m_0 = 0.5Mev$.

2. Griffiths 6.5. In this problem we have the harmonic oscillator problem with a perturbation of the form $H = -qEx$. The first order shift is simply $E_n^{(1)} = -qE\langle n|x|n\rangle = 0$. The second order shift is given by

$$\begin{aligned} E_n^{(2)} &= \sum_{m\neq n} q^2 E^2 \frac{|\langle m|x|n\rangle|^2}{E_n - E_m}, \\ &= q^2 E^2 \frac{\hbar}{2m\omega}\left(\frac{n}{\hbar\omega} + \frac{n+1}{-\hbar\omega}\right), \\ &= \frac{-(qE)^2}{2m\omega^2}. \end{aligned}$$

This problem can also be solved exactly, which is what is done in part (b). Using the change of variables suggested: $x' = x - (qE/m\omega^2)$, we expand the quadratic potential to see that $H(x) = H(x') - \text{constant}$. We know that the energies of $H(x')$ are just the usual $(n + 1/2)\hbar\omega$, evaluating the constant, we see that $\epsilon = \langle H(x)\rangle = \langle H(x')\rangle - (q^2E^2/(2m\omega^2)) = (n+1/2)\hbar\omega - (q^2E^2/(2m\omega^2))$.

3. Griffiths 6.12. We can write the problem out as follows

$$\begin{aligned}
\langle\frac{1}{r}\rangle &= -\frac{4\pi\epsilon_0}{e^2}\langle V\rangle, \\
&= -\frac{4\pi\epsilon_0}{e^2}\left(\frac{2}{n^2}\left(-\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right)\right), \text{ using the Virial Th,} \\
&= \frac{m}{n^2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right), \\
&= \frac{1}{n^2 a}.
\end{aligned}$$

4. Griffiths 6.14 In this problem we need to solve for

$$\begin{aligned}
E &= \frac{-1}{8m^3c^2}\langle n|p^4|n\rangle, \\
&= \frac{-m^2}{32m^3c^2}\langle n|(a_+ + a_-)^4|n\rangle.
\end{aligned}$$

By expanding out $(a_+ + a_-)^4$, and keeping only terms that have the same number of creation and annihilation operators, and using $a_+|n\rangle = i\sqrt{(n+1)\hbar\omega}|n+1\rangle$ and $a_-|n\rangle = -i\sqrt{n\hbar\omega}|n-1\rangle$, we find that

$$E = \frac{-3\hbar^2\omega^2}{32mc^2}(2n^2 + 2n + 1).$$

5. Griffiths 6.25. With a little bit of algebra, you can show that

$$\begin{aligned}
E_{FS} &= -\gamma\left(3 - \frac{8}{j + 1/2}\right), \\
E_Z &= \beta(m_l + 2m_s)
\end{aligned}$$

Following the choice of basis used by Griffiths, we use the $|jm_j\rangle$ basis in which the H_{FS} perturbation is diagonal. Also on pg. 281, Griffiths is kind enough to write out this basis in terms of the $|lm_l\rangle \otimes |sm_s\rangle$ basis in which our other perturbation H_z is diagonal. So first of all we can immediately write down the H_{FS} perturbation. For the $j = 1/2$ we have $E = -5\gamma$, and since we are calculating the $-W$ matrix, we find that for the 1st, 2nd, 6th and 8th diagonal element we get 5γ. The remaining diagonal elements have $j = 3/2$ giving us just γ for the $-W$ matrix. And so we are done with first perturbation. The second perturbation is diagonal for the first four wavefunctions since they are both eigenstates in the $|jm_j\rangle$ basis and the $|lm_l\rangle \otimes |sm_s\rangle$ basis. We read off the matrix elements as $-\beta(m_l + 2m_s) = -\beta, +\beta, -2\beta, +2\beta$ respectively. Notice that

$$
\begin{aligned}
-\langle\psi_5|L_z + 2S_z|\psi_5\rangle = -2/3\hbar\beta, \quad -\langle\psi_6|L_z + 2S_z|\psi_6\rangle = -1/3\hbar\beta \\
-\langle\psi_5|L_z + 2S_z|\psi_6\rangle = -\langle\psi_6|L_z + 2S_z|\psi_5\rangle = \sqrt{2}/3\hbar\beta \\
-\langle\psi_7|L_z + 2S_z|\psi_7\rangle = -2/3\hbar\beta, \quad -\langle\psi_8|L_z + 2S_z|\psi_8\rangle = -1/3\hbar\beta \\
-\langle\psi_8|L_z + 2S_z|\psi_7\rangle = -\langle\psi_7|L_z + 2S_z|\psi_8\rangle = \sqrt{2}/3\hbar\beta
\end{aligned}
$$

Putting all this together, we get the $-W$ matrix as required.

6. Griffiths 6.33. Suppose the Hamiltonian H, for a particular quantum system, is a function of some parameter λ; let $E_n(\lambda)$ and $\psi_n(\lambda)$ be the eigenvalues and eigenfunctions of $H(\lambda)$. The Feynman-Hellman theorem states that

$$\frac{\partial E_n}{\partial \lambda} = \left\langle \psi_n | \frac{\partial H}{\partial \lambda} | \psi_n \right\rangle$$

The effective Hamiltonian for the radial wave functions of hydrogen is

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2} - \frac{e^2}{4\pi\epsilon_0}\frac{1}{r},$$

and the eigenvalues are

$$E_n = -\frac{me^4}{32\pi^2\epsilon_0^2\hbar^2(j_{max} + l + 1)^2}.$$

(a) Use $\lambda = e$ in the Feynman-Hellmann theorem to obtain $\langle 1/r \rangle$. (Griffiths Equation 6.55)

[First we compute

$$\frac{\partial H}{\partial e} = \frac{-2e}{4\pi\epsilon_0}\frac{1}{r}$$

Then

$$\left\langle \psi_n \mid \frac{\partial H}{\partial e} \mid \psi_n \right\rangle \quad = \quad -2\frac{e}{4\pi\epsilon_0}\left\langle \psi_n \mid \frac{1}{r} \mid \psi_n \right\rangle$$

According to the Feynman-Hellmann theorem

$$\begin{aligned}
\frac{\partial E_n}{\partial e} &= -2\frac{e}{4\pi\epsilon_0}\left\langle \psi_n \mid \frac{1}{r} \mid \psi_n \right\rangle \\
\Rightarrow \frac{4}{e}E_n &= -2\frac{e}{4\pi\epsilon_0}\langle\frac{1}{r}\rangle \\
\Rightarrow \langle\frac{1}{r}\rangle &= -2\left(\frac{4\pi\epsilon_0}{e^2}\right)E_n \\
&= 2\left(\frac{4\pi\epsilon_0\hbar c}{e^2}\right)\frac{1}{n^2}\frac{1}{2}mc^2\frac{\alpha^2}{\hbar c} \\
&= 2\frac{1}{n^2}\frac{1}{2}\frac{mc^2\alpha}{\hbar c} \\
&= \frac{1}{an^2}
\end{aligned}$$

where $a = \hbar c/\alpha mc^2$ is the Bohr radius.]

(b) Use $\lambda = l$ to obtain $\langle 1/r^2\rangle$. (Griffiths Equation 6.56) [This time we have that

$$\begin{aligned}
\frac{\partial E_n}{\partial l} &= \langle\frac{\partial H}{\partial l}\rangle \\
\Rightarrow \frac{-2E_n}{j_{max}+l+1} &= \frac{(2l+1)\hbar^2}{2m}\langle\frac{1}{r^2}\rangle \\
\Rightarrow \langle\frac{1}{r^2}\rangle &= -2E_n\frac{2m}{(2l+1)\hbar^2}\frac{1}{j_{max}+l+1}
\end{aligned}$$

Since

$$E_n\frac{m}{\hbar^2} = -\frac{1}{n^2}\frac{1}{2}\alpha mc^2\frac{m}{\hbar^2} = -\frac{1}{2n^2}\frac{1}{a^2}$$

we have that

$$\langle\frac{1}{r^2}\rangle = \frac{1}{a^2}\frac{2}{(2l+1)}\frac{1}{(j_{max}+l+1)^3}]$$

7. Griffiths 6.36. In this problem we examine the Stark effect for $n = 1$ and $n = 2$. With the electric field in the z-direction, the Hamiltonian is

$$H_s = -eE_{ext}z = -eE_{ext}r\cos\theta,$$

We will treat this as a perturbation to the Bohr Hamiltonian.

(a) To first order, the change in the ground state is given by

$$
\begin{aligned}
E_s &= \langle 100|H_s|100\rangle, \\
&= (const)\int \exp(-2r/a)r^3 dr \int_{-1}^{1} \cos\theta \; d(\cos\theta) = 0
\end{aligned}
$$

(b) Lets define the states that we will work with

$$
\begin{aligned}
|1\rangle &= \psi_{200} = \sqrt{\frac{1}{2\pi a}\frac{1}{2a}}\left(1-\frac{r}{2a}\right)\exp\left(\frac{-r}{2a}\right), \\
|2\rangle &= \psi_{211} = -\sqrt{\frac{1}{\pi a}\frac{1}{8a^2}} r \exp\left(\frac{-r}{2a}\right)\sin\theta e^{i\phi}, \\
|3\rangle &= \psi_{210} = \sqrt{\frac{1}{2\pi a}\frac{1}{4a^2}} r \exp\left(\frac{-r}{2a}\right)\cos\theta, \\
|4\rangle &= \psi_{21-1} = \sqrt{\frac{1}{\pi a}\frac{1}{8a^2}} r \exp\left(\frac{-r}{2a}\right)\sin\theta e^{-i\phi}.
\end{aligned}
$$

We need to compute the W-matrix

$$
\begin{pmatrix} W_{11} & W_{12} & W_{13} & W_{14} \\ W_{21} & W_{22} & W_{23} & W_{24} \\ W_{31} & W_{32} & W_{33} & W_{34} \\ W_{41} & W_{42} & W_{43} & W_{44} \end{pmatrix} = \begin{pmatrix} \langle 1 \mid H'_S \mid 1\rangle & \langle 1 \mid H'_S \mid 2\rangle & \langle 1 \mid H'_S \mid 3\rangle & \langle 1 \mid H'_S \mid 4\rangle \\ \langle 2 \mid H'_S \mid 1\rangle & \langle 2 \mid H'_S \mid 2\rangle & \langle 2 \mid H'_S \mid 3\rangle & \langle 2 \mid H'_S \mid 4\rangle \\ \langle 3 \mid H'_S \mid 1\rangle & \langle 3 \mid H'_S \mid 2\rangle & \langle 3 \mid H'_S \mid 3\rangle & \langle 3 \mid H'_S \mid 4\rangle \\ \langle 4 \mid H'_S \mid 1\rangle & \langle 4 \mid H'_S \mid 2\rangle & \langle 4 \mid H'_S \mid 3\rangle & \langle 4 \mid H'_S \mid 4\rangle \end{pmatrix}
$$

Either by direct calculation or by inspection, convince yourself that the only non-zero terms are $\langle 1|H'_s|3\rangle = \langle 3|H'_s|1\rangle$, which we proceed to calculate.

$$
\begin{aligned}
\langle 1|H'_s|3\rangle &= -eE_{ext}\int \frac{1}{2\pi a}\frac{1}{8a^3}\left(1-\frac{r}{2a}\right) r \;\exp\left(\frac{-r}{a}\right)\cos\theta(r\cos\theta)r^2 \; dr \; d\Omega, \\
&= \frac{-eE_{ext}}{8a^4}\left(\int_0^\infty \left(1-\frac{r}{2a}\right) r^4 \exp(-r/a)\; dr\right)\left(\int_{-1}^{1}\cos^2\theta \; (d\cos\theta)\right), \\
&= -\frac{eaE_{ext}}{12}(\Gamma(5)-\Gamma(6)/2), \\
&= -3eaE_{ext}.
\end{aligned}
$$

Then

$$
W = -3eaE_{ext}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
$$

The eigenvalues are $\pm 3eaE_{ext}$ so when the perturbation is turned on the degeneracy is split into 3 different energies, $E_2^0, E_2^0 \pm 3eaE_{ext}$.

(c) The "good" wave functions are formed from the eigenvectors of the W-matrix

$$\psi_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} \quad \text{and} \quad \psi_3 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}$$

The four good wave functions are

$$\begin{aligned}
\psi_1 &= \frac{1}{\sqrt{2}}(\psi_{200} + \psi_{210}) \\
\psi_2 &= \psi_{211} \\
\psi_3 &= \frac{1}{\sqrt{2}}(\psi_{200} - \psi_{210}) \\
\psi_4 &= \psi_{21-1}
\end{aligned}$$

(d) The dipole moment $\mathbf{p}_e = -e\mathbf{r} = -er(\sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k})$. The ϕ integral for the expectation value of the x and y components will give zero for all four states. The expectation value of the z component can be constructed from the elements of the W matrix.

$$\begin{aligned}
\langle\psi_1 \mid \mathbf{p}_e \mid \psi_1\rangle &= \frac{1}{2}(W_{13} + W_{31}) = 3ae\hat{k} \\
\langle\psi_2 \mid \mathbf{p}_e \mid \psi_2\rangle &= 0 \\
\langle\psi_3 \mid \mathbf{p}_e \mid \psi_3\rangle &= -\frac{1}{2}(W_{13} + W_{31}) = -3ae\hat{k} \\
\langle\psi_4 \mid \mathbf{p}_e \mid \psi_4\rangle &= 0
\end{aligned}$$

8. Positronium. We can rewrite our two perturbations in a more transparent form. Using $S^2 = S_1^2 + S_2^2 + 2S_1.S_2$ and defining $\beta = eB/mc$, we have our two perturbations of the form:

$$\begin{aligned}
H_{HFS} &= \frac{\alpha}{2}\left(s(s+1) - \frac{3}{2}\right), \\
H_B &= \beta(s_{1z} - s_{2z}).
\end{aligned}$$

Following the method in the previous problem we have:

$$W = \left(\begin{array}{cccc|c} |11\rangle & |10\rangle & |1-1\rangle & |00\rangle & \\ \hline \alpha/4 & 0 & 0 & 0 & |11\rangle \\ 0 & \alpha/4 & 0 & \beta & |10\rangle \\ 0 & 0 & \alpha/4 & 0 & |1-1\rangle \\ 0 & \beta & 0 & -3\alpha/4 & |00\rangle \end{array}\right)$$

By diagonalizing this matrix we find the following

$$\begin{aligned} E(\psi_1) &= E_0 + \alpha/4 \\ E(\psi_3) &= E_0 + \alpha/4 \\ E(\psi_+) &= E_0 - \frac{\alpha}{4} + \frac{1}{2}\sqrt{\alpha^2 + 4\beta^2} \\ E(\psi_-) &= E_0 - \frac{\alpha}{4} - \frac{1}{2}\sqrt{\alpha^2 + 4\beta^2} \end{aligned}$$

Please check that in the limit $\alpha \to 0$, we get the correct splitting where the four fold degeneracy is broken by one level increasing by β and one decreasing by β, while the other two remain the same, while for the opposite limit $\beta \to 0$, we have the singlet-triplet splitting with three levels increasing in energy by $\alpha/4$ and the singlet shift its energy by $-3\alpha/4$.