QM1 Problem Set 1 solutions — Mike Saelim

If you find any errors with these solutions, please email me at mjs496@cornell.edu.

1 **(a)** We can assume that the function is smooth enough to Taylor expand:

$$f(A) = \sum_n c_n A^n.$$

The rest is a straightforward evaluation, using the property $A|a\rangle = a|a\rangle$:

$$\begin{aligned}\langle a|f(A)|a'\rangle &= \sum_n c_n \langle a|A^n|a'\rangle = \sum_n c_n \langle a|a'^n|a'\rangle \\ &= \sum_n c_n a'^n \langle a|a'\rangle = f(a')\delta_{aa'}.\end{aligned}$$

(b) Since the operator can be written as a diagonal matrix of its eigenvalues, $A = \sum_a a|a\rangle\langle a|$, the determinant is simply the product of all these eigenvalues.

The trace of the logarithm of the operator is

$$\mathrm{Tr}\ln A = \sum_a \langle a|\ln A|a\rangle = \sum_a \ln a,$$

so taking the exponential of that gives

$$\exp(\mathrm{Tr}\ln A) = \exp(\sum_a \ln a) = \prod_a a = \det A.$$

(c) e^{iA} is unitary if $e^{iA}(e^{iA})^\dagger = \mathbf{1}$, where $\mathbf{1}$ is the identity operator. To show this, let's decompose the exponentials into their spectral decompositions.

The spectral decomposition of any function $f(A)$ is $\sum_a f(a)|a\rangle\langle a|$, which can be reasoned out from what we proved in part (a). (You can also compute it out directly by replacing A with its spectral decomposition in the Taylor expansion of $f(A)$.) So,

$$\begin{aligned}e^{iA} &= \sum_a e^{ia}|a\rangle\langle a| \\ (e^{iA})^\dagger &= \sum_a (e^{ia}|a\rangle\langle a|)^\dagger = \sum_a e^{-ia}|a\rangle\langle a|.\end{aligned}$$

Putting these together,

$$\begin{aligned}e^{iA}(e^{iA})^\dagger &= \sum_a\sum_b e^{ia}e^{-ib}|a\rangle\langle a|b\rangle\langle b| = \sum_a\sum_b e^{ia}e^{-ib}|a\rangle\delta_{ab}\langle b| \\ &= \sum_a |a\rangle\langle a| = \mathbf{1}.\end{aligned}$$

Alternatively, given that we showed $(e^{iA})^\dagger = e^{-iA}$, we could just say that $e^{iA}(e^{iA})^\dagger = e^{iA}e^{-iA} = e^{\mathbf{0}} = \mathbf{1}$, but we would have to specify that we can only combine the two exponentials naively because their arguments commute. ($e^X e^Y = e^{X+Y}$ is not true if $[X,Y] \neq 0$.)

$\boxed{2}$ I find it easiest to start with the right-hand side and work to the left-hand side of this equation. We can even use our friends the Taylor expansion and spectral decomposition again.

$$
\begin{aligned}
f(U^{-1}AU) &= \sum_n c_n (U^{-1}AU)^n = \sum_n c_n U^{-1}AUU^{-1}AUU^{-1}AU \dots U^{-1}AU \\
&= U^{-1} \sum_n c_n A^n U = U^{-1} f(A) U.
\end{aligned}
$$

$\boxed{3}$ **(a)** Calculating $e^{i\hat{n}\cdot\vec{\sigma}\phi}$ involves using the Taylor expansion of the exponential, which will necessitate calculating $(\hat{n}\cdot\vec{\sigma})^2$. One way to do this is by brute force. If we let $\hat{n} = (n_x, n_y, n_z)$,

$$
\begin{aligned}
\hat{n}\cdot\vec{\sigma} &= \begin{pmatrix} n_z & n_x - in_y \\ n_x + in_y & -n_z \end{pmatrix} \\
(\hat{n}\cdot\vec{\sigma})^2 &= \begin{pmatrix} n_x^2 + n_y^2 + n_z^2 & 0 \\ 0 & n_x^2 + n_y^2 + n_z^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}
$$

since $\hat{n}$ is a unit vector. Thus, factors of $(\hat{n}\cdot\vec{\sigma})^n$ will simply alternate between the identity matrix and $\hat{n}\cdot\vec{\sigma}$.

The other way to do this is to cheat (sorta) and use the identity you prove in part (b).

$$
(\hat{n}\cdot\vec{\sigma})^2 = (\hat{n}\cdot\hat{n})\cdot\mathbf{1} + i\vec{\sigma}\cdot(\hat{n}\times\hat{n}) = \mathbf{1}.
$$

Either way,

$$
\begin{aligned}
e^{i(\hat{n}\cdot\vec{\sigma})\phi} &= \mathbf{1} + i(\hat{n}\cdot\vec{\sigma})\phi - \frac{1}{2}(\hat{n}\cdot\vec{\sigma})^2\phi^2 - \frac{i}{3!}(\hat{n}\cdot\vec{\sigma})^3\phi^3 + \dots \\
&= \mathbf{1} + i(\hat{n}\cdot\vec{\sigma})\phi - \frac{1}{2}\mathbf{1}\phi^2 - \frac{1}{3!}i(\hat{n}\cdot\vec{\sigma})^3\phi^3 + \dots \\
&= \cos\phi\cdot\mathbf{1} + i(\hat{n}\cdot\vec{\sigma})\sin\phi
\end{aligned}
$$

where the bold **1** signifies the 2x2 identity matrix.

(b) The quickest way to prove this identity is to use two identities that define the Pauli matrices:

$$
[\sigma_i, \sigma_j] = \sum_k 2i\varepsilon_{ijk}\sigma_k \qquad\qquad \{\sigma_i, \sigma_j\} = 2\delta_{ij}\cdot\mathbf{1}.
$$

Then,

$$
\begin{aligned}
(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) &= \sum_{ij} a_i b_j \sigma_i \sigma_j = \sum_{ij} a_i b_j \left(\frac{1}{2}\{\sigma_i, \sigma_j\} + \frac{1}{2}[\sigma_i, \sigma_j]\right) \\
&= \sum_{ij} a_i b_j \left(\delta_{ij}\cdot\mathbf{1} + \sum_k i\varepsilon_{ijk}\sigma_k\right) = \vec{a}\cdot\vec{b}\cdot\mathbf{1} + i\vec{\sigma}\cdot\vec{a}\times\vec{b}.
\end{aligned}
$$

$\boxed{4}$ **(a)** Again with the Taylor expansions:

$$
\begin{aligned}
\frac{de^{xB}}{dx} &= \frac{d}{dx}\left(1 + xB + \frac{1}{2}x^2B^2 + \frac{1}{3!}x^3B^3 + \dots\right) \\
&= B + xB^2 + \frac{1}{2}x^2B^3 + \dots = B\,e^{xB}.
\end{aligned}
$$

Or, this can also be done with the definition of the derivative, extended to functions of operators:

$$\begin{aligned}\frac{de^{xB}}{dx} &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}[e^{(x+\epsilon)B} - e^{xB}] \\ &= e^{xB}\lim_{\epsilon\to 0}\frac{1}{\epsilon}[e^{\epsilon B} - 1] \\ &= e^{xB}\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\epsilon B + \frac{1}{2}\epsilon^2 B^2 + \dots\right] \\ &= B\, e^{xB}\end{aligned}$$

but one must be careful to specify that you can split $e^{(x+\epsilon)B}$ into $e^{xB}e^{\epsilon B}$ only because xB and ϵB commute.

(b) This will require us to use the definition of the derivative, as well as a Taylor expansion of the operators $A(x)$ and $B(x)$:

$$\begin{aligned}\frac{dAB}{dx} &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}[A(x+\epsilon)B(x+\epsilon) - A(x)B(x)] \\ &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\left(A(x)+\epsilon\frac{dA(x)}{dx}+O(\epsilon^2)\right)\left(B(x)+\epsilon\frac{dB(x)}{dx}+O(\epsilon^2)\right) - A(x)B(x)\right] \\ &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[A(x)B(x)+\epsilon\frac{dA(x)}{dx}B(x)+\epsilon A(x)\frac{dB(x)}{dx}+O(\epsilon^2) - A(x)B(x)\right] \\ &= A'B + AB'.\end{aligned}$$

(c) This relies on the identity in part (b). Since $AA^{-1} = \mathbf{1}$,

$$\begin{aligned}\frac{d}{dx}(AA^{-1}) &= \frac{dA}{dx}A^{-1} + A\frac{dA^{-1}}{dx} = 0 \\ \implies \frac{dA^{-1}}{dx} &= -A^{-1}\frac{dA}{dx}A^{-1}.\end{aligned}$$

5 **Photon Filters**

The linear polarizer can be represented by a matrix that picks out the component of the photon's state vector that is parallel to the axis rotated by α from the x-axis. In the x-y basis, it takes the x and y components of the input state vector, and spits out the x and y components of the output state vector. We can do this by rotating the state vector by $-\alpha$, picking out its x-component, and rotating the result back by α:

$$P_\alpha^{xy} = \begin{pmatrix}\cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}1 & 0 \\ 0 & 0\end{pmatrix}\begin{pmatrix}\cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha\end{pmatrix} = \begin{pmatrix}\cos^2\alpha & \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & \sin^2\alpha\end{pmatrix}.$$

What does the matrix look like if we have it take in and spit out R and L components instead? One way is to transform P_α^{xy} with the matrices that transform the components of the state vector. The prompt gives us the transformation for the basis states:

$$\begin{pmatrix}|R\rangle \\ |L\rangle\end{pmatrix} = U\begin{pmatrix}|x\rangle \\ |y\rangle\end{pmatrix} \qquad U = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i \\ 1 & -i\end{pmatrix}.$$

However, the components of the state vector will actually transform with U^*:

$$
\begin{aligned}
|\xi\rangle = \begin{pmatrix}\xi_x & \xi_y\end{pmatrix}\begin{pmatrix}|x\rangle \\ |y\rangle\end{pmatrix} &= \begin{pmatrix}\xi_x & \xi_y\end{pmatrix} U^\dagger U \begin{pmatrix}|x\rangle \\ |y\rangle\end{pmatrix} = \begin{pmatrix}\xi_R & \xi_L\end{pmatrix}\begin{pmatrix}|R\rangle \\ |L\rangle\end{pmatrix} \\
\implies \begin{pmatrix}\xi_R & \xi_L\end{pmatrix} &= \begin{pmatrix}\xi_x & \xi_y\end{pmatrix} U^\dagger \\
\implies \begin{pmatrix}\xi_R \\ \xi_L\end{pmatrix} &= U^* \begin{pmatrix}\xi_x \\ \xi_y\end{pmatrix}
\end{aligned}
$$

So, to transform P_α into a matrix that transforms between R-L components, we use the U^* matrix that transforms x-y components into R-L components, and its inverse U^T that transforms R-L components into x-y components:

$$
\begin{aligned}
P_\alpha^{RL} &= U^*\, P_\alpha^{xy}\, U^T \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i \\ 1 & i\end{pmatrix}\begin{pmatrix}\cos^2\alpha & \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & \sin^2\alpha\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1 \\ i & -i\end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix}1 & e^{-2i\alpha} \\ e^{2i\alpha} & 1\end{pmatrix}.
\end{aligned}
$$

Another way, which pretty much does the same thing but avoids the confusion over what matrix to use, involves playing with bras and kets. Notice that $P_\alpha = |\alpha\rangle\langle\alpha|$ where $|\alpha\rangle = \cos\alpha|x\rangle + \sin\alpha|y\rangle$ in the x-y basis. We can transform this to the R-L basis:

$$
\begin{aligned}
|\alpha\rangle &= \cos\alpha\,\frac{1}{\sqrt{2}}(|R\rangle + |L\rangle) + \sin\alpha\,\frac{-i}{\sqrt{2}}(|R\rangle - |L\rangle) \\
&= \frac{1}{\sqrt{2}}(e^{-i\alpha}|R\rangle + e^{i\alpha}|L\rangle).
\end{aligned}
$$

Using this $|\alpha\rangle$, P_α gives the same result.

We can do a similar transformation from Q^{xy} to Q^{RL} to account for the behavior of the quarter-wave plate.

$$
Q^{xy} = \begin{pmatrix}1 & 0 \\ 0 & i\end{pmatrix} \qquad \implies \qquad Q^{RL} = U^*\, Q^{xy}\, U^T = \frac{1}{2}\begin{pmatrix}1+i & 1-i \\ 1-i & 1+i\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}e^{i\pi/4} & e^{-i\pi/4} \\ e^{-i\pi/4} & e^{i\pi/4}\end{pmatrix}.
$$

Now let's consider the combination $QP_{\pi/4}Q$. Plugging and chugging,

$$
\begin{aligned}
(Q\,P_{\pi/4}\,Q)^{xy} &= \frac{1}{2}\begin{pmatrix}1 & i \\ i & -1\end{pmatrix} \\
(Q\,P_{\pi/4}\,Q)^{RL} &= \begin{pmatrix}0 & 1 \\ 0 & 0\end{pmatrix}
\end{aligned}
$$

So, we can summarize the effects of this system in a table:

Input	Output	Transmission probability = $\lvert\text{Output}\rvert^2$
$\lvert x\rangle$	$\frac{1}{2}\lvert R\rangle$	$\frac{1}{2}$
$\lvert y\rangle$	$\frac{i}{2}\lvert R\rangle$	$\frac{1}{2}$
$\lvert R\rangle$	0	0
$\lvert L\rangle$	$\lvert R\rangle$	1

This filter blocks right-circularly polarized light, and turns left-circularly polarized light into right-circularly polarized light.

Indeed, this fits with what we would expect $Q\,P_{\pi/4}\,Q$ to do. We've set the fast and slow axes of our quarter wave plates to be parallel to the x and y axes, so light linearly polarized parallel to the x and y axes will be unaffected by the first quarter-wave plate: the plate cannot induce a shift between two components if only one is present. Then, it is halved in intensity by the 45° linear polarizer, and turned into circularly-polarized light by the second quarter-wave plate. The first quarter-wave plate will also shift left-circularly polarized light into linearly polarized light parallel to the axis of the linear polarizer, while it shifts right-circularly polarized light into linearly polarized light perpendicular to that axis.

6 Sakurai 1.7

Since we can decompose any state into a linear combination of eigenstates of A, $|\alpha\rangle = \sum_a c_a|a\rangle$, let's consider these operators' effects on eigenstates first.

(a) For some eigenstate $|a\rangle$,

$$\prod_{a'}(A-a')|a\rangle = \prod_{a'}(a-a')|a\rangle$$

which equals 0 because of the term $a'=a$. Since this holds for all the eigenstates, the operator will return 0 for any state.

(b) For some eigenstate $|a\rangle$,

$$\prod_{a''\neq a'}\frac{(A-a'')}{(a'-a'')}|a\rangle = \prod_{a''\neq a'}\frac{(a-a'')}{(a'-a'')}|a\rangle.$$

We have two possible cases: either $a=a'$ and all the terms will be 1, or $a\neq a'$, and one term will have 0 in the numerator. So,

$$\prod_{a''\neq a'}\frac{(A-a'')}{(a'-a'')} = |a'\rangle\langle a'|,$$

is the projection operator. Note that this is only possible if the states are nondegenerate, because if two different states have the same eigenvalue, the product will have 0/0 for one of its terms.

(c) The null operator,

$$\prod_{a'}(S_z-a') = \left(S_z+\frac{\hbar}{2}\right)\left(S_z-\frac{\hbar}{2}\right),$$

will return 0 for either state.

The projection operator for $S_z \to -\frac{\hbar}{2}$,

$$\prod_{a''\neq -\hbar/2}\frac{(S_z-a'')}{(-\frac{\hbar}{2}-a'')} = \frac{1}{2}-\frac{S_z}{\hbar},$$

will return $|{-\hbar/2}\rangle$ when acting on $|{-\hbar/2}\rangle$ and 0 when acting on $|\hbar/2\rangle$.