PHYS 6572 - Quantum Mechanics I - Fall 2011

Problem Set 7 — Solutions

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For your reference, here are some useful identities invoked frequently on this problem set:

$$\begin{aligned}
J^2|\,j,m\rangle &= j(j+1)\hbar^2|\,j,m\rangle \\
J_z|\,j,m\rangle &= m\hbar|\,j,m\rangle \\
J_\pm &= J_x \pm iJ_y \\
J_\pm|\,j,m\rangle &= \hbar\sqrt{(j\mp m)(j\pm m+1)}|\,j,m\pm 1\rangle
\end{aligned}$$

1 Angular momentum

(a). There are at least two ways to approach this problem:

- Using the ladder operators, $J_x = \frac{1}{2}(J_+ + J_-)$ and $J_y = \frac{1}{2i}(J_+ - J_-)$. When J_x (resp. J_y) acts on $|\,j,m\rangle$, it produces a linear combination of two ket states $|\,j,m+1\rangle$ and $|\,j,m-1\rangle$, both of which are orthogonal to $|\,j,m\rangle$. So if we put the bra $\langle j,m\,|$ with the ket $J_x|\,j,m\rangle$ (resp. $J_y|\,j,m\rangle$) together, the bracket must vanish: that is, the expectation value of J_x (resp. J_y) in the state $|\,j,m\rangle$ is 0.
- It is also possible to exploit the commutation relations of the J_i alone without invoking the ladder operators. Since $[J_y, J_z] = i\hbar J_x$, and $J_z|\,j,m\rangle = m\hbar|\,j,m\rangle$, we can compute the expectation value of J_x in $|\,j,m\rangle$ as follows:

$$\begin{aligned}
\langle j,m\mid J_x\mid j,m\rangle &= \frac{1}{i\hbar}\langle j,m\mid [J_y,J_z]\mid j,m\rangle \\
&= \frac{1}{i\hbar}\left(\langle j,m\mid J_yJ_z\mid j,m\rangle - \langle j,m\mid J_zJ_y\mid j,m\rangle\right) \\
&= \frac{1}{i\hbar}\left(m\hbar\,\langle j,m\mid J_y\mid j,m\rangle - m\hbar\,\langle j,m\mid J_y\mid j,m\rangle\right) \\
&= 0.
\end{aligned}$$

Essentially the same arguments go to show that $\langle j,m\mid J_y\mid j,m\rangle = 0$.

(b). For this part we use:

- The commutation relation $[J_i, J_j] = i\hbar\epsilon_{ijk}J_k$ $(i,j,k = 1, 2.3)$.
- The Jacobi identity of commutators (Lie brackets): $[A,[B,C]] = -([B,[C,A]]+[C,[A,B]])$.

Thus if an operator $\mathcal{O}$ commutes with both J_i and J_j $(i \neq j)$, then it must commute with the third component:

$$[\mathcal{O}, J_k] = \frac{\epsilon_{ijk}}{i\hbar}[\mathcal{O},[J_i,J_j]] = -\frac{\epsilon_{ijk}}{i\hbar}\left\{[J_i,\cancel{[J_j,\mathcal{O}]}] + [J_j,\cancel{[\mathcal{O},J_i]}]\right\} = 0.$$

2 Spin precession

We're given a spin-1/2 particle $|\,\psi\rangle$ in a B field

$$\mathbf{B}(t) = B\cos(\omega t)\hat{\mathbf{i}} + B\sin(\omega t)\hat{\mathbf{j}} + B_0\hat{\mathbf{k}}$$

which consists of a static z-component and an oscillating component in the xy-plane. In this lab frame, $|\,\psi\rangle$ evolves according to the Schrödinger equation

$$i\hbar\frac{d}{dt}|\,\psi(t)\rangle = H(t)|\,\psi(t)\rangle = -\gamma(\mathbf{S}\cdot\mathbf{B}(t))|\,\psi(t)\rangle.$$

Due to the time-dependence of $\mathbf{B}$ (or H), it is more complicated to write down the solution $|\,\psi(t)\rangle$ in the lab frame. So instead we shifts to a frame which co-rotates with the oscillating field, and introduce $|\,\psi_r(t)\rangle = e^{-i\omega S_z t/\hbar}|\,\psi(t)\rangle$, which should see a time-independent B field. What is the effective Hamiltonian H_r in the rotating frame? A direct calculation shows that

$$\begin{aligned}
i\hbar\frac{d}{dt}|\,\psi_r(t)\rangle &= i\hbar\frac{d}{dt}\left(e^{-i\omega S_z t/\hbar}|\,\psi(t)\rangle\right) \\
&= i\hbar\left(-i\omega S_z/\hbar\right)e^{-i\omega S_z t/\hbar}|\,\psi(t)\rangle + i\hbar e^{-i\omega S_z t/\hbar}\frac{d}{dt}|\,\psi(t)\rangle \\
&= \omega S_z|\,\psi_r(t)\rangle + e^{-i\omega S_z t/\hbar}H(t)|\,\psi(t)\rangle \\
&= \underbrace{\left[\omega S_z + e^{-i\omega S_z t/\hbar}H(t)e^{i\omega S_z t/\hbar}\right]}_{=H_r}|\,\psi_r(t)\rangle.
\end{aligned}$$

To go on we must unravel the operator $e^{-i\omega S_z t/\hbar}H(t)e^{i\omega S_z t/\hbar}$. This can be done by considering its matrix representation in the eigenbasis of S_z, $\{|\,+\rangle, |\,-\rangle\}$: Clearly

$$e^{-i\omega S_z t/\hbar} = \begin{pmatrix} e^{-i\omega t/2} & 0 \\ 0 & e^{i\omega t/2}\end{pmatrix} \quad \text{and} \quad e^{i\omega S_z t/\hbar} = \left(e^{-i\omega S_z t/\hbar}\right)^\dagger.$$

Meanwhile,

$$\begin{aligned}
H(t) = -\gamma(\mathbf{S}\cdot\mathbf{B}(t)) &= -\frac{\hbar\gamma}{2}\sum_{j=1}^{3}\sigma_j B_j \\
&= -\frac{\hbar\gamma}{2}\begin{pmatrix} B_0 & B[\cos(\omega t) + i\sin(\omega t)] \\ B[\cos(\omega t) - i\sin(\omega t)] & -B_0\end{pmatrix} \\
&= -\frac{\hbar\gamma}{2}\begin{pmatrix} B_0 & Be^{i\omega t} \\ Be^{-i\omega t} & -B_0\end{pmatrix}.
\end{aligned}$$

Thus

$$\begin{aligned}
e^{-i\omega S_z t/\hbar}H(t)e^{i\omega S_z t/\hbar} &= -\frac{\hbar\gamma}{2}\begin{pmatrix} e^{-i\omega t/2} & 0 \\ 0 & e^{i\omega t/2}\end{pmatrix}\begin{pmatrix} B_0 & Be^{i\omega t} \\ Be^{-i\omega t} & -B_0\end{pmatrix}\begin{pmatrix} e^{i\omega t/2} & 0 \\ 0 & e^{-i\omega t/2}\end{pmatrix} \\
&= -\frac{\hbar\gamma}{2}\begin{pmatrix} e^{-i\omega t/2} & 0 \\ 0 & e^{i\omega t/2}\end{pmatrix}\begin{pmatrix} B_0 e^{i\omega t/2} & Be^{i\omega t/2} \\ Be^{-i\omega t/2} & -B_0 e^{-i\omega t/2}\end{pmatrix} \\
&= -\frac{\hbar\gamma}{2}\begin{pmatrix} B_0 & B \\ B & -B_0\end{pmatrix} \\
&= -\gamma(B_0 S_z + BS_x).
\end{aligned}$$

Putting it together, we get

$$i\hbar\frac{d}{dt}|\,\psi_r(t)\rangle = -\gamma(\mathbf{S}\cdot\mathbf{B}_r)|\,\psi_r(t)\rangle \quad \text{where} \;\; \mathbf{B}_r = B\hat{\mathbf{i}}_r + \left(B_0 - \frac{\omega}{\gamma}\right)\hat{\mathbf{k}}.$$

Now that the effective Hamiltonian $H_r = -\gamma(\mathbf{S}\cdot\mathbf{B}_r)$ is time-independent, we may easily write down the time evolution of any state in the rotating frame as

$$|\,\psi_r(t)\rangle = e^{-iH_r t/\hbar}|\,\psi_r(0)\rangle = e^{i\gamma(\mathbf{S}\cdot\mathbf{B}_r)t/\hbar}|\,\psi_r(0)\rangle.$$

To explicitly compute the action of the unitary evolution operator $U(t) = e^{i\gamma(\mathbf{S}\cdot\mathbf{B}_r)t/\hbar}$ on an arbitrary state, it helps to exploit the following identities: Define

$$\begin{aligned} |\,\hat{\mathbf{n}};+\rangle &= \cos\left(\frac{\theta}{2}\right)|\,+\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|\,-\rangle \\ |\,\hat{\mathbf{n}};-\rangle &= -\sin\left(\frac{\theta}{2}\right)|\,+\rangle + e^{i\phi}\cos\left(\frac{\theta}{2}\right)|\,-\rangle, \end{aligned}$$

where $\hat{\mathbf{n}}$ is the unit vector in $\mathbb{R}^3$ with azimuthal angle θ (from $+\hat{\mathbf{k}}$) and polar angle ϕ. Then

$$(\mathbf{S}\cdot\hat{\mathbf{n}})|\,\hat{\mathbf{n}};\pm\rangle = \pm\frac{\hbar}{2}|\,\hat{\mathbf{n}};\pm\rangle.$$

In the current problem, the operator $\mathbf{S}\cdot\mathbf{B}_r = |\mathbf{B}_r|(\mathbf{S}\cdot\hat{\mathbf{n}})$, where

$$|\mathbf{B}_r| = \sqrt{B^2 + \left(B_0 - \frac{\omega}{\gamma}\right)^2} \;\; \text{and } \hat{\mathbf{n}} \text{ has associated angles } \theta = \sin^{-1}\left(\frac{B}{|\mathbf{B}_r|}\right),\;\; \phi = 0.$$

Therefore it has eigenkets $|\,\hat{\mathbf{n}};\pm\rangle$ with eigenvalues $\pm|\mathbf{B}_r|\hbar/2$. By the functional calculus of operators (see #1, PS1), the operator $e^{i\gamma(\mathbf{S}\cdot\mathbf{B}_r)t/\hbar}$ has the same eigenkets $|\,\hat{\mathbf{n}};\pm\rangle$ with eigenvalues $e^{\pm i\gamma|\mathbf{B}_r|t/2}$. This means that

$$|\,\psi_r(t)\rangle = e^{i\gamma|\mathbf{B}_r|t/2}\langle\hat{\mathbf{n}};+\,|\,\psi_r(0)\rangle|\,\hat{\mathbf{n}};+\rangle + e^{-i\gamma|\mathbf{B}_r|t/2}\langle\hat{\mathbf{n}};-\,|\,\psi_r(0)\rangle|\,\hat{\mathbf{n}};-\rangle.$$

Now suppose the initial ket is $|\,\psi_r(0)\rangle = |\,\psi(0)\rangle = |\,+\rangle$, per the problem. Then in the rotating frame its time evolution is given by

$$\begin{aligned} |\,\psi_r(t)\rangle &= e^{i\gamma|\mathbf{B}_r|t/2}\langle\hat{\mathbf{n}};+\,|\,+\rangle|\,\hat{\mathbf{n}};+\rangle + e^{-i\gamma|\mathbf{B}_r|t/2}\langle\hat{\mathbf{n}};-\,|\,+\rangle|\,\hat{\mathbf{n}};-\rangle \\ &= e^{i\gamma|\mathbf{B}_r|t/2}\cos\left(\frac{\theta}{2}\right)|\,\hat{\mathbf{n}};+\rangle - e^{-i\gamma|\mathbf{B}_r|t/2}\sin\left(\frac{\theta}{2}\right)|\,\hat{\mathbf{n}};-\rangle \\ &= e^{i\gamma|\mathbf{B}_r|t/2}\cos\left(\frac{\theta}{2}\right)\left[\cos\left(\frac{\theta}{2}\right)|\,+\rangle + \sin\left(\frac{\theta}{2}\right)|\,-\rangle\right] \\ &\quad - e^{-i\gamma|\mathbf{B}_r|t/2}\sin\left(\frac{\theta}{2}\right)\left[-\sin\left(\frac{\theta}{2}\right)|\,+\rangle + \cos\left(\frac{\theta}{2}\right)|\,-\rangle\right] \\ &= \left[\cos\left(\frac{\gamma|\mathbf{B}_r|t}{2}\right) + i\sin\left(\frac{\gamma|\mathbf{B}_r|t}{2}\right)\cos\theta\right]|\,+\rangle + i\sin\left(\frac{\gamma|\mathbf{B}_r|t}{2}\right)\sin\theta|\,-\rangle \\ &= \left[\cos\left(\frac{\omega_r t}{2}\right) + i\left(\frac{\omega_0-\omega}{\omega_r}\right)\sin\left(\frac{\omega_r t}{2}\right)\right]|\,+\rangle + i\left(\frac{\gamma B}{\omega_r}\right)\sin\left(\frac{\omega_r t}{2}\right)|\,-\rangle, \end{aligned}$$

where we have used the shorthands $\omega_0 = B_0/\gamma$ and $\omega_r = |\mathbf{B}_r|/\gamma$.

Back in the lab frame, the state would read

$$\begin{aligned} |\,\psi(t)\rangle &= e^{i\omega S_z t/\hbar}|\,\psi_r(t)\rangle \\ &= \left[\cos\left(\frac{\omega_r t}{2}\right) + i\left(\frac{\omega_0-\omega}{\omega_r}\right)\sin\left(\frac{\omega_r t}{2}\right)\right] e^{i\omega S_z t/\hbar}|\,+\rangle + i\left(\frac{\gamma B}{\omega_r}\right)\sin\left(\frac{\omega_r t}{2}\right) e^{i\omega S_z t/\hbar}|\,-\rangle \\ &= \left[\cos\left(\frac{\omega_r t}{2}\right) + i\left(\frac{\omega_0-\omega}{\omega_r}\right)\sin\left(\frac{\omega_r t}{2}\right)\right] e^{i\omega t/2}|\,+\rangle + i\left(\frac{\gamma B}{\omega_r}\right)\sin\left(\frac{\omega_r t}{2}\right) e^{-i\omega t/2}|\,-\rangle. \end{aligned}$$

Note that $|\,\psi(0)\rangle = |\,+\rangle$, as required. It follows that $\langle S_z(0)\rangle = \langle\psi(0)\mid S_z\mid\psi(0)\rangle = \hbar/2$.

If $\omega=\omega_0$ (”on resonance”), the effective field $\mathbf{B}_r$ in the rotating frame consists of the transverse component $B\hat{\mathbf{i}}_r$ only, so the spin state precesses about the $\hat{\mathbf{i}}_r$ axis at angular frequency $\omega_r=\omega_0=\gamma B$. From the perspective of the lab frame, the state evolves as

$$\begin{aligned} |\,\psi(t)\rangle &= \cos\left(\frac{\omega_0 t}{2}\right) e^{i\omega_0 t/2}|\,+\rangle + i\sin\left(\frac{\omega_0 t}{2}\right) e^{-i\omega_0 t/2}|\,-\rangle \\ &= e^{i\omega_0 t/2}\left[\cos\left(\frac{\omega_0 t}{2}\right)|\,+\rangle + e^{i\pi/2}\sin\left(\frac{\omega_0 t}{2}\right) e^{-i\omega_0 t}|\,-\rangle\right] \\ &= e^{i\omega_0 t/2}|\,\hat{\mathbf{n}}(t);+\rangle, \end{aligned}$$

where $\hat{\mathbf{n}}(t)$ is the unit vector with associated angles $\theta(t)=\omega_0 t$ and $\phi(t)=(\pi/2)-\omega_0 t$. The orientations of the spin state trace out a figure-8 on the Bloch sphere (Fig. 1). Note that the up state $|\,+\rangle$ flips into the down state $|\,-\rangle$ in a duration $T=\pi/\omega_0$, and vice versa.

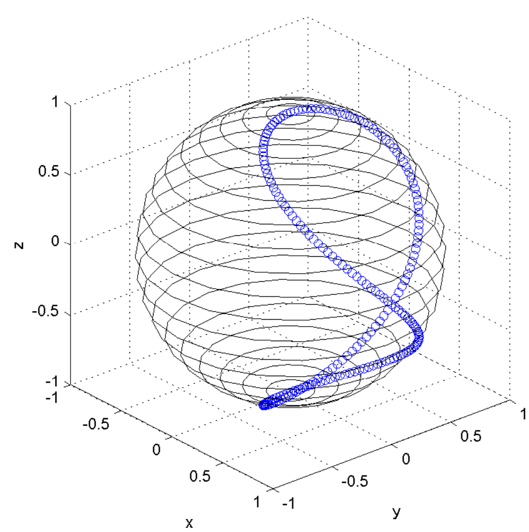

Figure 1: Time evolution of a spin-1/2 state in a field $\mathbf{B}(t) = B\cos(\omega t)\hat{\mathbf{i}} + B\sin(\omega t)\hat{\mathbf{j}} + B_0\hat{\mathbf{k}}$ when on resonance ($\omega=\gamma B_0$). Initial state is the ”up” state $|\,+\rangle = |\,\hat{\mathbf{z}};+\rangle$. Dots on the Bloch sphere indicate the successive spin orientations in the lab frame.

For general ω, the z-magnetization of the state $|\,\psi(t)\rangle$ at time t is

$$\begin{aligned} &\langle S_z(t)\rangle \\ =\ &\langle\psi(t)\mid S_z\mid\psi(t)\rangle \end{aligned}$$

$$\begin{aligned}
&= \left(\left[\cos\left(\frac{\omega_r t}{2}\right) - i\left(\frac{\omega_0-\omega}{\omega_r}\right)\sin\left(\frac{\omega_r t}{2}\right)\right]e^{-i\omega t/2}\langle +| - i\left(\frac{\gamma B}{\omega_r}\right)\sin\left(\frac{\omega_r t}{2}\right)e^{i\omega t/2}\langle -|\right) \\
&\quad \times S_z\left(\left[\cos\left(\frac{\omega_r t}{2}\right) + i\left(\frac{\omega_0-\omega}{\omega_r}\right)\sin\left(\frac{\omega_r t}{2}\right)\right]e^{i\omega t/2}|+\rangle + i\left(\frac{\gamma B}{\omega_r}\right)\sin\left(\frac{\omega_r t}{2}\right)e^{-i\omega t/2}|-\rangle\right) \\
&= \frac{\hbar}{2}\left[\left|\cos\left(\frac{\omega_r t}{2}\right) + i\left(\frac{\omega_0-\omega}{\omega_r}\right)\sin\left(\frac{\omega_r t}{2}\right)\right|^2 - \left(\frac{\gamma B}{\omega_r}\right)^2\sin^2\left(\frac{\omega_r t}{2}\right)\right] \\
&= \frac{\hbar}{2}\left[\cos^2\left(\frac{\omega_r t}{2}\right) + \frac{(\omega_0-\omega)^2-(\gamma B)^2}{\omega_r^2}\sin^2\left(\frac{\omega_r t}{2}\right)\right] \\
&= \frac{\hbar}{2}\left[\frac{1+\cos(\omega_r t)}{2} + \frac{(\omega_0-\omega)^2-(\gamma B)^2}{\omega_r^2}\left(\frac{1-\cos(\omega_r t)}{2}\right)\right] \\
&= \frac{\hbar}{2}\left[\frac{2(\omega_0-\omega)^2}{2\omega_r^2} + \frac{2(\gamma B)^2}{2\omega_r^2}\cos(\omega_r t)\right] \\
&= \langle S_z(0)\rangle\left[\frac{(\omega_0-\omega)^2}{(\omega_0-\omega)^2+(\gamma B)^2} + \frac{(\gamma B)^2}{(\omega_0-\omega)^2+(\gamma B)^2}\cos(\omega_r t)\right].
\end{aligned}$$

From this we deduce that the up state reverses orientation in a duration

$$T = \frac{\pi}{\omega_r} = \frac{\pi}{\sqrt{(\omega_0-\omega)^2+(\gamma B)^2}}.$$

3 Angular momentum of an unknown particle (Sakurai 3.15)

(a). It helps to rewrite $\psi(\mathbf{x})$ in spherical coordinates:

$$\psi(\mathbf{x}) = rf(r)(\sin\theta\cos\phi + \sin\theta\sin\phi + 3\cos\theta).$$

To check whether ψ is an eigenfunction of $\mathbf{L}^2$, we may carry out a direct computation. First note that the operator $\mathbf{L}^2$ can be explicitly written in spherical coordinates [*e.g.* Sakurai Eq. (3.6.15)]:

$$\mathbf{L}^2\psi(\mathbf{x}) = -\hbar^2\left[\frac{1}{\sin^2\theta}\partial_{\phi\phi} + \frac{1}{\sin\theta}\partial_\theta[(\sin\theta)\partial_\theta]\right]\psi(\mathbf{x})$$

$$\begin{aligned}
\partial_{\phi\phi}\psi(\mathbf{x}) &= -rf(r)(\sin\theta\cos\phi + \sin\theta\sin\phi) \\
\partial_\theta\psi(\mathbf{x}) &= rf(r)(\cos\theta\cos\phi + \cos\theta\sin\phi - 3\sin\theta) \\
\partial_\theta\left[(\sin\theta)\partial_\theta\right]\psi(\mathbf{x}) &= rf(r)\partial_\theta\left[\sin\theta\cos\theta(\cos\phi+\sin\phi) - 3\sin^2\theta\right] \\
&= rf(r)[\cos(2\theta)(\cos\phi+\sin\phi) - 3\sin(2\theta)]
\end{aligned}$$

So

$$\begin{aligned}
\mathbf{L}^2\psi(\mathbf{x}) &= -\hbar^2\left[\frac{1}{\sin^2\theta}[-\sin\theta(\cos\phi+\sin\phi)] + \frac{1}{\sin\theta}[\cos(2\theta)(\cos\phi+\sin\phi) - 3\sin(2\theta)]\right]rf(r) \\
&= -2\hbar^2 rf(r)\left[-\sin\theta(\cos\phi+\sin\phi) - 3\cos\theta\right] \\
&= 2\hbar^2\psi(\mathbf{x}).
\end{aligned}$$

Thus $\psi(\mathbf{x})$ is an eigenfunction of $\mathbf{L}^2$ with eigenvalue $l(l+1)\hbar^2 = 2\hbar^2$, or $l = 1$.

(b). Alternatively, we can re-express $\psi(\mathbf{x})$ as a linear combination of spherical harmonics. Using the normalized spherical harmonics

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta\ ,\quad Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi},$$

we find

$$\begin{aligned}\psi(\mathbf{x}) &= rf(r)\left[\sin\theta\left(\frac{e^{i\phi}+e^{-i\phi}}{2}+\frac{e^{i\phi}-e^{-i\phi}}{2i}\right)+3\cos\theta\right]\\ &= rf(r)\left[\frac{1}{2}(1-i)\sin\theta e^{i\phi}+\frac{1}{2}(1+i)\sin\theta e^{-i\phi}+3\cos\theta\right]\\ &= rf(r)\left[-\sqrt{\frac{2\pi}{3}}(1-i)Y_1^1+\sqrt{\frac{2\pi}{3}}(1+i)Y_1^{-1}+2\sqrt{3\pi}Y_1^0\right].\end{aligned}$$

In one fell swoop, we've shown that $\psi(\mathbf{x})$ is an eigenfunction of $\mathbf{L}^2$ with eigenvalue $1(1+1)\hbar^2 = 2\hbar^2$, and expanded $\psi(\mathbf{x})$ in the eigenbasis of the $j=1$ Hilbert space, *i.e,.* $\psi(\mathbf{x}) = rf(r)\sum_{m=-1}^{1} c_m Y_1^m$ where

$$c_1 = -\sqrt{\frac{2\pi}{3}}(1-i)\ ,\quad c_0 = 2\sqrt{3\pi}\ ,\quad c_{-1} = \sqrt{\frac{2\pi}{3}}(1+i).$$

It ought to be clear that the probability of ψ being found in the state $\mid 1, m\rangle$ is given by

$$P(m) = \frac{|c_m|^2}{\sum_{m=-1}^{1}|c_m|^2}.$$

Since $|c_0|^2 = 9|c_1|^2 = 9|c_{-1}|^2$, we have

$$P(1) = \frac{1}{11}\ ,\quad P(0) = \frac{9}{11}\ ,\quad P(-1) = \frac{1}{11}.$$

(c). Recall that the Laplacian in $\mathbb{R}^3$ can be written as

$$\Delta = \frac{1}{r^2}\partial_r\left(r^2\partial_r\right)+\frac{1}{r^2}\left[\frac{1}{\sin^2\theta}\partial_{\phi\phi}+\frac{1}{\sin\theta}\partial_\theta\left((\sin\theta)\partial_\theta\right)\right] = \frac{1}{r^2}\left[\partial_r\left(r^2\partial_r\right)-\frac{\mathbf{L}^2}{\hbar^2}\right].$$

So the time-independent Schrödinger equation in $\mathbb{R}^3$ takes the form

$$\left\{-\frac{\hbar^2}{2mr^2}\partial_r\left(r^2\partial_r\right)+\frac{\mathbf{L}^2}{2mr^2}+V(r)\right\}\Psi(\mathbf{x}) = E\Psi(\mathbf{x}). \tag{1}$$

Now suppose the energy eigenstate $\Psi(\mathbf{x})$ is the known wavefunction $\psi(\mathbf{x}) = rf(r)\zeta(\Omega)$, where $\zeta(\Omega) = \sin\theta\cos\phi+\sin\theta\sin\phi+3\cos\theta$. From Part (a) we already saw that $\mathbf{L}^2\psi(\mathbf{x}) = 2\hbar^2\psi(\mathbf{x})$. Meanwhile,

$$\begin{aligned}\partial_r\psi(\mathbf{x}) &= [f(r)+rf'(r)]\zeta(\Omega)\\ \partial_r(r^2\partial_r\psi(\mathbf{x})) &= 2r[f(r)+rf'(r)]\zeta(\Omega)+r^2[2f'(r)+rf''(r)]\zeta(\Omega)\\ &= r[2f(r)+4rf'(r)+r^2f''(r)]\zeta(\Omega).\end{aligned}$$

Plugging the various terms into (1) yields

$$-\frac{\hbar^2}{2mr}[\cancel{2f(r)}+4rf'(r)+r^2f''(r)]\zeta(\Omega)+\cancel{\frac{\hbar^2}{mr^2}[rf(r)\zeta(\Omega)]}+V(r)[rf(r)\zeta(\Omega)] = E[rf(r)\zeta(\Omega)].$$

Thus

$$V(r) = E+\frac{1}{rf(r)}\frac{\hbar^2}{2mr}\left[4rf'(r)+r^2f''(r)\right] = E+\frac{\hbar^2}{2mr^2}\left[\frac{4rf'(r)+r^2f''(r)}{f(r)}\right].$$

4 Rotated angular momentum (Sakurai 3.?)

A state $|\,\psi\rangle$ rotated by an angle β about the y-axis becomes $e^{-iJ_y\beta/\hbar}|\,\psi\rangle$. So the probability for the new state to be in $|\,2, m'\rangle$ ($m' = 0, \pm1, \pm2$) is given by the modulus squared of the projection of $e^{-iJ_y\beta/\hbar}|\,l = 2, m = 0\rangle$ onto the subspace $|\,l = 2, m'\rangle$, *i.e.*,

$$\left|\mathcal{D}^{(2)}_{m'0}(\alpha = 0, \beta, \gamma = 0)\right|^2 = \left|\left\langle 2, m' \mid e^{-iJ_y\beta/\hbar} \mid 2, 0\right\rangle\right|^2,$$

where α, β, and γ are the Euler angles. At this stage we may invoke Sakurai Eq. (3.6.52):[1]

$$\mathcal{D}^{(l)}_{m0}(\alpha, \beta, \gamma = 0) = \sqrt{\frac{4\pi}{2l+1}}\,Y_l^{m*}(\beta, \alpha).$$

Using the expressions $Y_2^m(\theta, \phi)$ in Appendix A, we find

$$\begin{aligned}
\mathcal{D}^{(2)}_{2,0}(\alpha = 0, \beta, \gamma = 0) &= \sqrt{\frac{4\pi}{5}}Y_2^{2*}(\beta, 0) = \sqrt{\frac{4\pi}{5}}\sqrt{\frac{15}{32\pi}}(\sin^2\beta) = \sqrt{\frac{3}{8}}\sin^2\beta,\\
\mathcal{D}^{(2)}_{1,0}(\alpha = 0, \beta, \gamma = 0) &= \sqrt{\frac{4\pi}{5}}Y_2^{1*}(\beta, 0) = \sqrt{\frac{4\pi}{5}}\left[-\sqrt{\frac{15}{8\pi}}(\sin\beta\cos\beta)\right] = -\sqrt{\frac{3}{2}}(\sin\beta\cos\beta),\\
\mathcal{D}^{(2)}_{0,0}(\alpha = 0, \beta, \gamma = 0) &= \sqrt{\frac{4\pi}{5}}Y_2^{0*}(\beta, 0) = \sqrt{\frac{4\pi}{5}}\sqrt{\frac{5}{16\pi}}(3\cos^2\beta - 1) = \frac{1}{2}(3\cos^2\beta - 1),\\
\mathcal{D}^{(2)}_{-1,0}(\alpha = 0, \beta, \gamma = 0) &= -\mathcal{D}^{(1)}_{1,0}(\alpha = 0, \beta, \gamma = 0),\\
\mathcal{D}^{(2)}_{-2,0}(\alpha = 0, \beta, \gamma = 0) &= \mathcal{D}^{(1)}_{2,0}(\alpha = 0, \beta, \gamma = 0).
\end{aligned}$$

Thus

$$\begin{aligned}
\left|\mathcal{D}^{(2)}_{\pm2,0}(\alpha = 0, \beta, \gamma = 0)\right|^2 &= \frac{3}{8}\sin^4\beta,\\
\left|\mathcal{D}^{(2)}_{\pm1,0}(\alpha = 0, \beta, \gamma = 0)\right|^2 &= \frac{3}{2}\sin^2\beta\cos^2\beta,\\
\left|\mathcal{D}^{(2)}_{0,0}(\alpha = 0, \beta, \gamma = 0)\right|^2 &= \frac{1}{4}(3\cos^2\beta - 1)^2.
\end{aligned}$$

It is straightforward to check that $\sum_{m=-2}^{2}\left|\mathcal{D}^{(2)}_{m0}(\alpha = 0, \beta, \gamma = 0)\right|^2 = 1$.

5 Rotation matrix for $j = 1$ states (Sakurai 3.22)

(a). Since $J_y = \frac{1}{2i}(J_+ - J_-)$, it is clear that the matrix element $\langle j, m' \mid J_y \mid j, m\rangle$ vanishes for any m, m' where $|m - m'| \neq 1$. Also $\langle j, m \mid J_y \mid j, m'\rangle = (\langle j, m' \mid J_y \mid j, m\rangle)^*$ by hermiticity of J_y. So for $j = 1$, it is enough to compute the matrix elements $\langle 1, 0 \mid J_y \mid 1, 1\rangle$ and $\langle 1, 0 \mid J_y \mid 1, -1\rangle$. From

$$\begin{aligned}
J_y|\,1, 1\rangle &= \frac{1}{2i}(\cancel{J_+|\,1, 1\rangle} - J_-|\,1, 1\rangle) = -\frac{1}{2i}(\sqrt{2}\hbar)|\,1, 0\rangle = \frac{i\hbar}{\sqrt{2}}|\,1, 0\rangle;\\
J_y|\,1, -1\rangle &= \frac{1}{2i}(J_+|\,1, -1\rangle - \cancel{J_-|\,1, -1\rangle}) = \frac{1}{2i}(\sqrt{2}\hbar)|\,1, 0\rangle = -\frac{i\hbar}{\sqrt{2}}|\,1, 0\rangle,
\end{aligned}$$

[1] Please read Sakurai Eqs. (3.6.46) through (3.6.51) and the accompanying text for the derivation.

we deduce that $\langle 1,0 \mid J_y \mid 1,1\rangle = (i\hbar)/\sqrt{2}$ and $\langle 1,0 \mid J_y \mid 1,-1\rangle = -(i\hbar)/\sqrt{2}$. So the matrix representation of J_y in the $\{\mid 1,1\rangle, \mid 1,0\rangle, \mid 1,-1\rangle\}$ basis reads

$$J_y^{(j=1)} = \begin{pmatrix} 0 & -\frac{i\hbar}{\sqrt{2}} & 0 \\ \frac{i\hbar}{\sqrt{2}} & 0 & -\frac{i\hbar}{\sqrt{2}} \\ 0 & \frac{i\hbar}{\sqrt{2}} & 0 \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 0 & -\sqrt{2}i & 0 \\ \sqrt{2}i & 0 & -\sqrt{2}i \\ 0 & \sqrt{2}i & 0 \end{pmatrix}.$$

(b). A direct computation shows that

$$[J_y^{(j=1)}]^2 = \left(\frac{\hbar}{2}\right)^2 \begin{pmatrix} 0 & -\sqrt{2}i & 0 \\ \sqrt{2}i & 0 & -\sqrt{2}i \\ 0 & \sqrt{2}i & 0 \end{pmatrix} \begin{pmatrix} 0 & -\sqrt{2}i & 0 \\ \sqrt{2}i & 0 & -\sqrt{2}i \\ 0 & \sqrt{2}i & 0 \end{pmatrix} = \left(\frac{\hbar}{2}\right)^2 \begin{pmatrix} 2 & 0 & -2 \\ 0 & 4 & 0 \\ -2 & 0 & 2 \end{pmatrix}$$

and

$$[J_y^{(j=1)}]^3 = \left(\frac{\hbar}{2}\right)^3 \begin{pmatrix} 0 & -\sqrt{2}i & 0 \\ \sqrt{2}i & 0 & -\sqrt{2}i \\ 0 & \sqrt{2}i & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & -2 \\ 0 & 4 & 0 \\ -2 & 0 & 2 \end{pmatrix} = \frac{\hbar^3}{2} \begin{pmatrix} 0 & -\sqrt{2}i & 0 \\ \sqrt{2}i & 0 & -\sqrt{2}i \\ 0 & \sqrt{2}i & 0 \end{pmatrix}.$$

In other words, $\left[J_y^{(j=1)}/\hbar\right]^3 = J_y^{(j=1)}/\hbar$, which means that for positive integers n,

$$\left[J_y^{(j=1)}/\hbar\right]^n = \begin{cases} \left[J_y^{(j=1)}/\hbar\right], & n \text{ odd} \\ \left[J_y^{(j=1)}/\hbar\right]^2, & n \text{ even} \end{cases}.$$

Therefore

$$\begin{aligned} e^{-iJ_y^{(1)}\beta/\hbar} &= \mathbf{1} + \sum_{n=0}^{\infty} \frac{(-i\beta)^{2n+1}}{(2n+1)!} \left(\frac{J_y^{(1)}}{\hbar}\right)^{2n+1} + \sum_{n=1}^{\infty} \frac{(-i\beta)^{2n}}{(2n)!} \left(\frac{J_y^{(1)}}{\hbar}\right)^{2n} \\ &= \mathbf{1} - i\sum_{n=0}^{\infty} \frac{(-1)^n\beta^{2n+1}}{(2n+1)!} \left(\frac{J_y^{(1)}}{\hbar}\right) + \sum_{n=1}^{\infty} \frac{(-1)^n\beta^{2n}}{(2n)!} \left(\frac{J_y^{(1)}}{\hbar}\right)^2 \\ &= \mathbf{1} - i\left(\frac{J_y^{(1)}}{\hbar}\right)\sin\beta + \left(\frac{J_y^{(1)}}{\hbar}\right)^2 (\cos\beta - 1). \end{aligned}$$

(c). The matrix representation $d^{(1)}(\beta)$ of $e^{-iJ_y^{(1)}\beta/\hbar}$ reads

$$\begin{aligned} d^{(1)}(\beta) &= \mathbf{1} - \frac{i\sin\beta}{2} \begin{pmatrix} 0 & -\sqrt{2}i & 0 \\ \sqrt{2}i & 0 & -\sqrt{2}i \\ 0 & \sqrt{2}i & 0 \end{pmatrix} + (\cos\beta - 1)\left(\frac{1}{2}\right)^2 \begin{pmatrix} 2 & 0 & -2 \\ 0 & 4 & 0 \\ -2 & 0 & 2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}(1+\cos\beta) & -\frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1-\cos\beta) \\ \frac{1}{\sqrt{2}}\sin\beta & \cos\beta & -\frac{1}{\sqrt{2}}\sin\beta \\ \frac{1}{2}(1-\cos\beta) & \frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1+\cos\beta) \end{pmatrix}. \end{aligned}$$

6 Neutrino oscillations

By assumption, the initial state of the neutrino is the weak eigenstate

$$|\,\psi(0)\rangle = |\,\nu_e\rangle = \cos\theta|\,\nu_1\rangle + \sin\theta|\,\nu_2\rangle,$$

where $|\,\nu_j\rangle$ ($j = 1, 2$) are the mass eigenstates, and θ is the mixing angle. Since the neutrinos are assumed free, the mass eigenstates evolves in time according to

$$|\,\nu_j(t)\rangle = e^{-iHt/\hbar}|\,\nu_j(0)\rangle = e^{-iE_jt/\hbar}|\,\nu_j(0)\rangle \quad \text{where} \;\; E_j = \sqrt{(m_jc^2)^2 + (pc)^2}.$$

The assumption that $|\,\nu_e\rangle$ is a momentum eigenstate allows us to replace the operator $\hat{p}$ with the scalar p. As a result, $|\,\psi\rangle$ evolves in time according to

$$\begin{aligned} |\,\psi(t)\rangle &= e^{-iHt/\hbar}|\,\psi(0)\rangle \\ &= e^{-iE_1t/\hbar}\langle\nu_1\,|\,\psi(0)\rangle|\,\nu_1\rangle + e^{-iE_2t/\hbar}\langle\nu_2\,|\,\psi(0)\rangle|\,\nu_2\rangle \\ &= e^{-iE_1t/\hbar}\cos\theta|\,\nu_1\rangle + e^{-iE_2t/\hbar}\sin\theta|\,\nu_2\rangle. \end{aligned}$$

Thus the probability of the system being in $|\,\nu_\mu\rangle$ at time t is

$$\begin{aligned} |\langle\nu_\mu\,|\,\psi(t)\rangle|^2 &= \left|(-\sin\theta\langle\nu_1\,| + \cos\theta\langle\nu_2\,|)\left(e^{-iE_1t/\hbar}\cos\theta|\,\nu_1\rangle + e^{-iE_2t/\hbar}\sin\theta|\,\nu_2\rangle\right)\right|^2 \\ &= (\sin\theta\cos\theta)^2\left|-e^{-iE_1t/\hbar} + e^{-iE_2t/\hbar}\right|^2 \\ &= \sin^2(2\theta)\left|ie^{-i(E_1+E_2)t/(2\hbar)}\left[\frac{-e^{-i(E_1-E_2)t/(2\hbar)} + e^{i(E_1-E_2)t/(2\hbar)}}{2i}\right]\right|^2 \\ &= \sin^2(2\theta)\sin^2\left(\frac{(E_1-E_2)t}{2\hbar}\right). \end{aligned}$$

Knowing that the mass of the neutrino is very small, *i.e.*, $m_jc^2 \ll pc$, we may approximate E_j in the usual way:

$$E_j = \sqrt{(m_jc^2)^2 + (pc)^2} = pc\sqrt{1 + \left(\frac{m_jc^2}{pc}\right)^2} = pc\left[1 + \frac{1}{2}\left(\frac{m_jc^2}{pc}\right)^2 + \mathcal{O}\left(\left[\frac{m_jc^2}{pc}\right]^4\right)\right].$$

So

$$E_1 - E_2 = pc\left[\frac{1}{2}\frac{(m_1^2 - m_2^2)c^4}{(pc)^2} + \mathcal{O}\left(\left[\frac{m_jc^2}{pc}\right]^4\right)\right] = \frac{(m_1^2 - m_2^2)c^4}{2pc} + (\text{higher-order terms}).$$

Using the shorthand $\Delta(m^2) = m_1^2 - m_2^2$, we deduce that

$$|\langle\nu_\mu\,|\,\psi(t)\rangle|^2 \approx \sin^2(2\theta)\sin^2\left(\frac{\Delta(m^2)c^3t}{4p\hbar}\right),$$

which shows that neutrino oscillation occurs at period $T = (4\pi p\hbar)/(\Delta(m^2)c^3)$.