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Tensor Operators and the Wigner Eckart Theorem

Vector operator

The ket $\mid \alpha\rangle$ transforms under rotation to $\mid \alpha'\rangle = \mathcal{D}(R) \mid \alpha\rangle$. The expectation value of a vector operator in the rotated system is related to the expectation value in the original system as

$$\langle \alpha' \mid V_i \mid \alpha' \rangle = \left\langle \alpha \mid \mathcal{D}^\dagger V_i \mathcal{D} \mid \alpha \right\rangle = R_{ij} \left\langle \alpha \mid V_j \mid \alpha \right\rangle$$

With $\mathcal{D}(R) = e^{-\frac{i}{\hbar}\mathbf{J}\cdot\hat{\mathbf{n}}\theta}$, and R_{ij} an orthogonal 3X3 rotation matrix. Define a vector operator as an object that transforms according to

$$\mathcal{D}^\dagger V_i \mathcal{D} = R_{ij} V_j.$$

In the case of an infinitesimal rotation

$$R\mathbf{V} = \mathbf{V} + \delta\theta \hat{\mathbf{n}} \times \mathbf{V}.$$

The effect of an infinitesimal rotation of $\mathbf{V}$ about $\mathbf{J} \cdot \hat{\mathbf{n}}$ by $\delta\theta$ gives us

$$\begin{aligned} (1 + \frac{i}{\hbar}\mathbf{J}\cdot\mathbf{n}\delta\theta)\mathbf{V}(1 - \frac{i}{\hbar}\mathbf{J}\cdot\hat{\mathbf{n}}\delta\theta) &= \mathbf{V} + \delta\theta\hat{\mathbf{n}} \times \mathbf{V} \\ \rightarrow \frac{i}{\hbar}[\mathbf{J}\cdot\hat{\mathbf{n}}, \mathbf{V}] &= \hat{\mathbf{n}} \times \mathbf{V} \end{aligned}$$

Then

$$\begin{aligned} \frac{i}{\hbar}[J_i, V_j]\delta\theta &= \sum_j \epsilon_{ijk} V_k \\ \rightarrow [V_j, J_i] &= -i\hbar\epsilon_{ijk} V_k \\ \rightarrow [V_i, J_k] &= -i\hbar\epsilon_{ijk} V_j \\ \rightarrow [V_i, J_j] &= i\hbar\epsilon_{ijk} V_j \end{aligned}$$

$$[V_x, J_z] = -i\hbar V_y$$

or more generally

$$[V_i, J_j] = i\hbar\epsilon_{ijk}V_k$$

We can take that last as the definition of a vector operator. A rotation of a vector operator is accomplished by computing

$$\mathcal{D}^\dagger \mathbf{V}\mathcal{D}$$

It is convenient if we can write the operator in a basis of angular momentum eigenstates since we know how to write $\mathcal{D}$ in that basis. So we might write the position operator as

$$\begin{aligned} x &= \frac{r}{2}\sqrt{\frac{8\pi}{3}}(Y_1^1 + Y_1^{-1}) \\ y &= \frac{r}{2i}\sqrt{\frac{8\pi}{3}}(Y_1^1 - Y_1^{-1}) \\ z &= r\sqrt{\frac{4\pi}{3}}Y_1^0 \end{aligned} \qquad \begin{aligned} Y_1^0 &= \sqrt{\frac{3}{4\pi}}\frac{z}{r} \\ Y_1^{\pm 1} &= \mp\sqrt{\frac{3}{4\pi}}\frac{x \pm iy}{\sqrt{2}r} \end{aligned}$$

where $r = \sqrt{x^2 + y^2 + z^2}$. Some examples of vector operators are momentum, position, and angular momentum. If we write our vector using spherical harmonics as a basis then our definition of a vector operator reads

$$\mathcal{D}^\dagger V_m \mathcal{D} = \mathcal{D}^{*1}_{mm'} V_{m'}$$

where $m = \pm 1, 0$.

Cartesian tensor operator

We can form a rank 2 cartesian tensor T by taking the product of two cartesian vector operators U and V.

$$T_{i,j} = V_i U_j$$

Then the transformation property follows as

$$\mathcal{D}^\dagger T_{ij}\mathcal{D} = R_{ik}V_k R_{jl}V_l = R_{ik}R_{jl}V_k U_l$$

There are 9 components of this second rank cartesian tensor. And the components do not transform irreducibly. In particular we can write T as

$$T_{ij} = \delta_{ij}\frac{\mathbf{V}\cdot\mathbf{U}}{3} + (\frac{V_iU_j - V_jU_i}{2}) + (\frac{V_iU_j + V_jU_i}{2} - \delta_{ij}\frac{\mathbf{V}\cdot\mathbf{U}}{3})$$

Under rotations, the first term has a single component and is invariant, the second term has 3 components and transforms like a vector, and the third term is a symmetric, traceless tensor with 5 independent terms.

Meanwhile, suppose that $T = x_i p_j$ where x and p are position and momentum operators. The tensor will include a scalar $\mathbf{x} \cdot \mathbf{p}$ a vector $\mathbf{x} \times \mathbf{p}$ and a symmetric traceless tensor $\frac{1}{2}(x_i p_j + x_j p_i) - \frac{1}{3}\mathbf{x} \cdot \mathbf{p}$. We could write the operator in a spherical basis using spherical harmonics where we let

$$Y_1^{\pm 1} = \sqrt{\frac{3}{8\pi}} \frac{x \pm iy}{r} = \sqrt{\frac{3}{8\pi}} \sin\theta e^{\pm i\phi}, \qquad Y_1^0 = \sqrt{\frac{1}{4\pi}} \frac{z}{r} = \sqrt{\frac{1}{4\pi}} \cos\theta$$

and then

$$Y_1^{\pm 1}(V) = \sqrt{\frac{3}{8\pi}}(V_x \pm iV_y) = |V|\sqrt{\frac{3}{8\pi}} \sin\theta e^{\pm i\phi}, \qquad Y_1^0(V) = \sqrt{\frac{1}{4\pi}} V_z = V\sqrt{\frac{1}{4\pi}} \cos\theta$$

So we have a recipe for translating a cartesian vector into a spherical $l = 1$ tensor. Then we can combine the $l = 1$ tensors using Clebsch Gordon coefficients

$$T_q^k = \sum_{q_1, q_2} \langle kql_1l_2 \mid l_1q_1l_2q_2 \rangle E_{q_1}^{l_1} F_{q_2}^{l_2}$$

pretty much the same as

$$Y_l^m = \sum_{m_1, m_2} \langle l, ml_1, l_2 \mid l_1, m_1, l_2, m_2 \rangle Y_{l_1}^{m_1} Y_{l_2}^{m_2}$$

And how does a spherical harmonic transform under rotations? First let's start with an example. Let's suppose that we have a cartesian vector operator $V = (V_x, V_y, V_z)$. We can also write it as $V_{\pm 1} = \frac{1}{2}(V_x \pm iV_y), V_0 = V_z$. Let's consider an infinitesimal rotation of the vector about the z-axis. We will rotate the cartesian version using an orthogonal matrix (SO(3)) and the spherical version using a representation of SU(2). The SO(3) rotation of the cartesian vector is here

$$\mathbf{V}' = R_z(\epsilon)\mathbf{V} \sim \begin{pmatrix} 1 & -\epsilon & 0 \\ \epsilon & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \begin{pmatrix} V_x - \epsilon V_y \\ \epsilon V_x + V_y \\ V_z \end{pmatrix} \tag{1}$$

The SU(2) rotation of the spherical vector is

$$\mathbf{V}' = \mathcal{D}_z(\epsilon)\mathbf{V} \sim (1 - \frac{i}{\hbar}\epsilon J_z) \begin{pmatrix} V_1 \\ V_0 \\ V_{-1} \end{pmatrix} = \begin{pmatrix} 1 - i\epsilon & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 + i\epsilon \end{pmatrix} \begin{pmatrix} V_1 \\ V_0 \\ V_{-1} \end{pmatrix} = \begin{pmatrix} V_1(1 - i\epsilon) \\ V_0 \\ V_{-1}(1 + i\epsilon) \end{pmatrix}$$

Converting $V'_{spherical}$ to $V'_{cartesian}$ we get

$$V'_{cart} = \begin{pmatrix} V'_1 + V'_{-1} \\ -i(V'_1 - V'_{-1}) \\ V_0 \end{pmatrix} = \begin{pmatrix} V_1 + V_{-1} - i\epsilon(V_1 - V_{-1}) \\ -i(V_1 - V_{-1} - i\epsilon(V_1 + V_{-1})) \\ V_0 \end{pmatrix} = \begin{pmatrix} V_x + \epsilon V_y \\ V_y - \epsilon V_x \\ V_z \end{pmatrix} \quad (2)$$

Comparing Equation 1 with 2 we see that the SU(2) representation of the rotation by ϵ is $\mathcal{D}(-\epsilon)$ when the SO(3) representation is $R_z(\epsilon)$. We see this again as follows

$$\begin{aligned} |\,\hat{\mathbf{n}}'\rangle &= \mathcal{D}(R)|\,\hat{\mathbf{n}}\rangle \\ \mathcal{D}(R^{-1})|\,l,m\rangle &= \sum_{m'} |\,l,m'\rangle \mathcal{D}^l_{m'm}(R^{-1}) \\ \left\langle \hat{\mathbf{n}} \mid \mathcal{D}(R^{-1}) \mid l,m \right\rangle &= \sum_{m'} \langle \hat{\mathbf{n}} \mid l,m'\rangle \mathcal{D}^l_{m'm}(R^{-1}) \\ Y_l^m(\hat{\mathbf{n}}') &= \sum_{m'} Y_l^{m'}(\hat{\mathbf{n}})\mathcal{D}^l_{m'm}(R^{-1}) \end{aligned}$$

We can turn $Y_l^m(\hat{\mathbf{n}}')$ into a more general vector operator $\mathbf{V}$ by setting $V_\pm = |V|Y_1^{\pm 1}$ and $V_0 = |V|Y_1^0$.

$$\begin{aligned} \rightarrow \mathcal{D}^\dagger(R)Y_l^m(\mathbf{V})\mathcal{D}(R) &= \sum_{m'} Y_l^{m'}(\mathbf{V})\mathcal{D}^l_{m'm}(R^{-1}) \\ &= \sum_{m'} Y_l^{m'}(\mathbf{V})\mathcal{D}^{*l}_{mm'}(R) \end{aligned}$$

Now we define a spherical tensor as an object that transforms according to

$$\mathcal{D}^\dagger T_q^k \mathcal{D} = \sum_{m'} \mathcal{D}_{m'q}(R^{-1})T^k_{m'} = \sum_{m'} \mathcal{D}^{*k}_{qm'}(R)T^k_{m'}$$

On the left we rotate the state by R and then measure by taking the expectation value. On the right we first measure the expectation value of each component of the operator and then rotate those expectation values by R^{-1}. Then an infinitesimal rotation gives us

$$\begin{aligned} (1 + \frac{i}{\hbar}\mathbf{J}\cdot\hat{\mathbf{n}}\theta)T_q^k(1 - \frac{i}{\hbar}\mathbf{J}\cdot\mathbf{n}\theta) &= \sum_{m'} \left\langle km' \mid (1 + \frac{i}{\hbar}\mathbf{J}\cdot\hat{\mathbf{n}}) \mid kq \right\rangle T^k_{m'} \\ &= T_q^k + \frac{i}{\hbar}\sum_{m'} \langle km' \mid \mathbf{J}\cdot\hat{\mathbf{n}}\theta \mid kq\rangle T^k_{m'} \\ \rightarrow [\mathbf{J}\cdot\hat{\mathbf{n}}\theta, T_q^k] &= \sum_{m'} \langle km' \mid \mathbf{J}\cdot\hat{\mathbf{n}}\theta \mid kq\rangle T^k_{m'} \end{aligned}$$

If $\hat{n} = \hat{z}$ then

$$[J_z, T_q^k] = q\hbar T_q^k \tag{3}$$

or if $\hat{n} = \frac{1}{\sqrt{2}}(\hat{n}_x \pm i\hat{n}_y)$,

$$[J_\pm, T_q^k] = \hbar \sum_{m'} \sqrt{k(k+1) - q(q \pm 1)} \langle km' \mid kq \pm 1 \rangle T_{m'}^k = \hbar \sqrt{k(k+1) - q(q \pm 1)} T_{q\pm 1}^k \tag{4}$$

Selection Rule

We show that

$$\left\langle \alpha', j', m' \mid T_q^k \mid \alpha, j, m \right\rangle = 0, \quad \text{unless } m' = q + m$$

Using Equation 3 we have

$$\begin{aligned} 0 &= \left\langle \alpha', j', m' \mid [J_z, T_q^k] - \hbar q T_q^k \mid \alpha, j, m \right\rangle \\ &= \hbar \left\langle \alpha', j', m' \mid T_q^k (m' - m - q \mid \alpha, j, m \right\rangle \\ &= \hbar \left\langle \alpha', j', m' \mid T_q^k \mid \alpha, j, m \right\rangle (m' - m - q) \end{aligned}$$

so if $(m' - m - q) \neq 0$ then the expectation value of T_q^k does. If T_q^k us a vector operator ($k = 1$), then the matrix element $\left\langle \alpha', j', m' \mid T_q^k \mid \alpha, j, m \right\rangle$ is zero unless $m' - m = \pm 1, 0$ and $|j' - j| = 1, 0$.

Tensor operators transform under rotations the same as angular momentum eigenkets. From Equation 4 we see that

$$T_q^k \mid jm \rangle = \propto \mid j', q + m \rangle.$$

The effect of the operator is to add q units of angular momentum.

Combining Rotation matrices

We can use the Clebsch Gordan matrix to combine rotation matrices. Let's go back to the general transformation from the m_1, m_2 basis to the j, m basis.

$$\mid j, m \rangle = \sum_{m_1, m_2} \mid j_1, m_1, j_2, m_2 \rangle \langle j_1, j_2, m_1, m_2 \mid j, m \rangle$$

We can rotate the pieces by applying the rotation operator and we have

$$\mathcal{D}(R)|\, j,m\rangle = \sum \mathcal{D}_1(R)\mathcal{D}_2(R)|\, j_1,m_1,j_2,m_2\rangle\langle j_1,m_1,j_2,m_2 \mid j,m\rangle$$

or we could write

$$\mathcal{D}(R)|\, j,m\rangle = \sum \mathcal{D}_1(R)\mathcal{D}_2(R)|\, j_1,m_1\rangle|\, j_2,m_2\rangle\langle j_1,m_1,j_2,m_2 \mid j,m\rangle$$

Then multiply from the left by

$$\langle j,m \mid = \sum_{m_1',m_2'} \langle j_1,m_1',j_2,m_2' \mid\langle j_1,j_2,m_1',m_2' \mid j,m'\rangle$$

and we get

$$\mathcal{D}^{j}_{m',m} = \sum_{m_1',m_2'}\sum_{m_1,m_2} \mathcal{D}^{j_1}_{m_1',m_1}\mathcal{D}^{j_2}_{m_2',m_2}\langle j_1,j_2,m_1',m_2' \mid j,m\rangle\langle j_1,m_1,j_2,m_2 \mid j,m\rangle \qquad (5)$$

Clebsch-Gordan Series

If we invert the above we have

$$\mid j_1,m_1,j_2,m_2\rangle = \sum_{j,m} \mid j,m\rangle\langle j,m \mid j_1,m_1,j_2,m_2\rangle$$

Then rotation and multiplication from the left by the dual gives

$$\mathcal{D}^{j_1}_{m_1',m_1}\mathcal{D}^{j_2}_{m_2',m_2} = \sum_{j,m,j',m'} \langle j',m' \mid \mathcal{D} \mid j,m\rangle \, \langle j_1,m_1',j_2,m_2' \mid j',m'\rangle\langle j,m \mid j_1,m_1,j_2,m_2\rangle$$

The rotation does not change j so

$$\mathcal{D}^{j_1}_{m_1',m_1}\mathcal{D}^{j_2}_{m_2',m_2} = \sum_{j,m,m'} \mathcal{D}^{j}_{m',m}\langle j_1,m_1',j_2,m_2' \mid j,m'\rangle\langle j,m \mid j_1,m_1,j_2,m_2\rangle \qquad (6)$$

This is called the Clebsch Gordan series or the Kronecker or direct product of representations and we can write

$$D^{j_1} \otimes D^{j_1} = D^{j_1+j_2} \oplus D^{j_1+j_2-1} \oplus ...D^{j_1-j_2}$$

or

$$D^{j_1} \otimes D^{j_1} = \begin{pmatrix} D^{j_1+j_2} & 0 & \cdots & 0 \\ 0 & D^{j_1+j_2-1} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & D^{j_1-j_2} \end{pmatrix} = \mid 00\rangle\langle 00 \mid \alpha\rangle$$

Integration over rotations

We want to consider

$$\int dR\mathcal{D}(R)|\,\alpha\rangle = \sum_{j,m}\int dR\;\mathcal{D}(R)|\,j,m\rangle\langle j,m\mid\alpha\rangle$$

where

$$\int dR = \int_0^{4\pi}\frac{d\alpha}{4\pi}\int_0^{4\pi}\frac{d\gamma}{4\pi}\int_0^{\pi}\frac{d\beta\sin\beta}{2} = 1$$

Then

$$\int dR\;\mathcal{D}^j_{mm'} = \delta_{m,0}\delta_{m',0}\delta_{j,0}$$

This follows because integration over all angles averages over all directions and the average is zero except for the that state that has no direction, namely $j = m = 0$. Now we can use this result along with Equation 6 and write

$$\begin{aligned}\int \mathcal{D}^{j_1}_{m_1'm_1}\mathcal{D}^{j_2}_{m_2'm_2}\;dR &= \int dR\sum_{j,m,m'}\mathcal{D}^j_{m',m}\langle j_1,m_1',j_2,m_2'\mid j,m'\rangle\langle j,m\mid j_1,m_1,j_2,m_2\rangle\\ &= \langle j_1,m_1',j_2,m_2'\mid 0,0\rangle\langle 0,0\mid j_1,m_1,j_2,m_2\rangle\end{aligned}$$

We can guess at the relevant Clebsch Gordon coefficients. Write

$$|\,0,0,j_1,j_2\rangle = \sum_{m_1,m_2}\langle j_1,m_1,j_2,m_2\mid 0,0,j_1,j_2\rangle|\,j_1,m_1,j_2,m_2\rangle$$

It is clear that j_1 and j_2 must be equal and $m_1 = -m_2$ for the coefficients to be non zero. If for example $j_1 = j_2 = 2$ then

$$|\,0,0,2,2\rangle = a|\,2,2,2,-2\rangle + b|\,2,1,2,-1\rangle + c|\,2,0,2,0\rangle + d|\,2,-1,2,-1\rangle + e|\,2,-2,2,2\rangle$$

No matter how we rotate the state there will be equal parts of all five components so it must be that all the coefficients are equal magnitude. Then for the normalization to be right

$$\langle j,m,j,-m\mid 0,0,j,j\rangle = \frac{1}{\sqrt{2j+1}}$$

and we can write

$$\begin{aligned}\int \mathcal{D}^{j_1}_{m_1'm_1}\mathcal{D}^{j_2}_{m_2'm_2}\;dR &= \langle j_1,m_1',j_2,m_2'\mid 0,0\rangle\langle 0,0\mid j_1,m_1,j_2,m_2\rangle\\ &= \frac{\delta_{j_1,j_2}\delta_{m_1,-m_2}\delta_{m_1',-m_2'}}{2j+1}\end{aligned}$$

Then using $\mathcal{D}(R^{-1}) = \mathcal{D}^\dagger(R)$ we can see that

$$\mathcal{D}^j_{m,m'} = (-1)^{m-m'}\mathcal{D}^{*j}_{-m,-m'}$$

and we can write

$$\int \mathcal{D}^{j_1}_{m'_1 m_1}\mathcal{D}^{*j_2}_{m'_2 m_2}\, dR \;\; = \;\; \frac{\delta_{j_1,j_2}\delta_{m_1,m_2}\delta_{m'_1,m'_2}}{2j+1} \tag{7}$$

Integration of a triple product of rotation matrices

Begin with the Clebsch Gordon series, Equation 6.

$$\mathcal{D}^{j_1}_{m'_1,m_1}\mathcal{D}^{j_2}_{m'_2,m_2} = \sum_{j,m,m'}\mathcal{D}^j_{m',m}\langle j_1,m'_1,j_2,m'_2 \mid j,m'\rangle\langle j,m \mid j_1,m_1,j_2,m_2\rangle$$

and multiply both sides by ${\mathcal{D}^J_{MM'}}^*$ and integrate over R and use Equation 7

$$\begin{aligned}
\int dR\, \mathcal{D}^{j_1}_{m'_1,m_1}\mathcal{D}^{j_2}_{m'_2,m_2}{\mathcal{D}^J_{MM'}}^* &= \int dR \sum_{j,m,m'}\mathcal{D}^j_{m',m}\langle j_1,m'_1,j_2,m'_2 \mid j,m'\rangle\langle j,m \mid j_1,m_1,j_2,m_2\rangle{\mathcal{D}^J_{MM'}}^* \\
&= \sum_{j,m,m'}\langle j_1,m'_1,j_2,m'_2 \mid j,m'\rangle\langle j,m \mid j_1,m_1,j_2,m_2\rangle\frac{\delta_{m',M}\delta_{m,M'}\delta_{j,J}}{2j+1} \\
&= \langle j_1,m'_1,j_2,m'_2 \mid J,M\rangle\langle J,M' \mid j_1,m_1,j_2,m_2\rangle\frac{1}{2J+1} \qquad (8)
\end{aligned}$$

where we have used

$$\mathcal{D}^j_{m,m'}(R)^* = (-1)^{m-m'}\mathcal{D}^j_{-m,-m'}(R).$$

Meanwhile we have already shown that

$$\mathcal{D}^l_{m0}(\alpha,\beta,\gamma) = \sqrt{\frac{4\pi}{2l+1}}Y^*_{lm}(\beta,\alpha) = \sqrt{\frac{4\pi}{2l+1}}Y^*_{lm}(\mathbf{n})$$

Substitution into 8 gives

$$\int d\Omega Y^*_{LM}(\Omega)Y_{l_1,m_1}(\Omega)Y_{l_2,m_2}(\Omega) = \sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2L+1)}}\langle l_1,m_1,l_2,m_2 \mid LM\rangle\langle l_1,0,l_2,0 \mid L0\rangle$$

Wigner Eckart Theorem

Spherical tensor operators transform according to

$$\mathcal{D}^\dagger T_q^k \mathcal{D} \qquad = \sum_{q'} \mathcal{D}^{*k}{}_{q,q'} T_{q'}^k$$

which implies that

$$\begin{aligned}
T_q^k \qquad &= \sum_{q'} \mathcal{D}\mathcal{D}^{*k}{}_{q,q'} T_{q'}^k \mathcal{D}^\dagger \\
\rightarrow \quad & \left\langle \alpha, j, m \mid T_q^k \mid \alpha', j', m' \right\rangle \\
&= \sum_{q', m_1, m_2} \langle \alpha, j, m \mid \mathcal{D} \mid \alpha, j, m_1 \rangle \mathcal{D}_{q,q'}^{*k} \left\langle \alpha, j, m_1 \mid T_{q'}^k \mid \alpha', j', m_2 \right\rangle \left\langle \alpha', j', m_2 \mid \mathcal{D}^\dagger \mid \alpha', j', m' \right\rangle \\
&= \sum_{q', m_1, m_2} \mathcal{D}_{mm_1}^j \mathcal{D}_{q,q'}^{*k} \left\langle \alpha, j, m_1 \mid T_{q'}^k \mid \alpha', j', m_2 \right\rangle \mathcal{D}^{\dagger j'}{}_{m_2 m'}
\end{aligned}$$

Before integrating let's look at that more carefully.

$$\begin{aligned}
\mathcal{D}^\dagger T_q^k \mathcal{D} &= \sum_{q'} \mathcal{D}^{*k}{}_{qq'} T_{q'}^k \\
\langle jm | \mathcal{D}^\dagger T_q^k \mathcal{D} | j'm' \rangle &= \sum_{q'} \mathcal{D}^{*k}{}_{qq'} \langle jm | T_{q'}^k | j'm' \rangle \\
\sum_{m_1 m_2} \langle jm | \mathcal{D}^\dagger | jm_1 \rangle \langle jm_1 | T_q^k | j'm_2 \rangle \langle j'm_2 | \mathcal{D} | j'm' \rangle &= \sum_{q'} \mathcal{D}^{*k}{}_{qq'} \langle jm | T_{q'}^k | j'm' \rangle \\
\sum_{m_1 m_2} \mathcal{D}^{\dagger j}{}_{mm_1} \langle jm_1 | T_q^k | j'm_2 \rangle \mathcal{D}_{m_2 m'}^{j'} &= \sum_{q'} \mathcal{D}^{*k}{}_{qq'} \langle jm | T_{q'}^k | j'm' \rangle \\
\langle jm_1 | T_q^k | j'm_2 \rangle &= \sum_{q', m, m'} \mathcal{D}^{\dagger -1 j}{}_{m_1 m} \mathcal{D}^{*k}{}_{qq'} \langle jm | T_{q'}^k | j'm' \rangle \mathcal{D}^{-1 j'}{}_{m' m_2} \\
&= \sum_{q', m, m'} \mathcal{D}_{m_1 m}^j \mathcal{D}^{*k}{}_{qq'} \langle jm | T_{q'}^k | j'm' \rangle \mathcal{D}^{*j'}{}_{m_2 m'}
\end{aligned}$$

Then integrate over R using Equation 8 and we get

$$\begin{aligned}
\int dR \, \left\langle \alpha, j, m_1 \mid T_q^k \mid \alpha', j', m_2 \right\rangle &= \int dR \sum_{q', m, m'} \mathcal{D}_{m_1 m}^j \mathcal{D}^{*k}{}_{qq'} \langle \alpha jm | T_{q'}^k | \alpha' j'm' \rangle \mathcal{D}^{*j'}{}_{m_2 m'} \\
&= \frac{1}{2j+1} \sum_{q', m, m'} \langle \alpha jm | T_{q'}^k | \alpha' j'm' \rangle \langle jm_1 \mid kqj'm_2 \rangle \langle kq'j'm' \mid jm \rangle
\end{aligned}$$

$$\left\langle \alpha, j, m_1 \mid T_q^k \mid \alpha', j', m_2' \right\rangle \;=\; \frac{\langle jm_1 \mid k, q, j', m_2 \rangle}{2j+1} \sum_{q', m, m'} \langle \alpha j m \,|T_{q'}^k|\, \alpha' j' m' \rangle \langle k q' j' m' \mid jm \rangle$$

With the sum over q', m and m' we can rewrite that last equation as

$$\left\langle \alpha, j, m_1 \mid T_q^k \mid \alpha', j', m_2' \right\rangle \;=\; \frac{\langle jm_1 \mid k, q, j', m_2 \rangle}{\sqrt{2j+1}} \langle \alpha j \,||T^k||\, \alpha' j' \rangle$$

All of the remaining dependence on m_2, q, and m_1 is in the Clebsch Gordon coefficient $\langle j, m_1 \mid k, q, j', m_2 \rangle$. The so called reduced matrix element $\langle \alpha, k \,||T^k||\, \alpha', j' \rangle$ does not depend on m_1, m_2 or q. Perhaps the significance is more obvious if we write it as

$$\left\langle \alpha, j, m_1 \mid T_q^k \mid \alpha', j', m_2' \right\rangle \;=\; \frac{\langle jm_1 \mid k, q, j', m_2 \rangle}{\sqrt{2j+1}} c_{j,j'}(\alpha, \alpha')$$

$c_j(\alpha)$ is a number that depends only on j, j' and α. α represents aspects of the state that do not depend on orientation, like the radial dependence of a wave function. So for an initial and final state with j, j', α, α' and a spherical tensor operator T_q^k, once we have computed $\left\langle \alpha, j, m_1 \mid T_q^k \mid \alpha', j', m_2' \right\rangle$ for a particular m_1, m_2 and q, we can with the help of the Clebsch Gordon coefficient on the right, determine the reduced matrix element $c_{j,j'\alpha,\alpha'}$. Then we compute any of the others by multiplying by the appropriate Clebsch Gordon coefficient.

Tensor operator recursion relationship

$$\begin{aligned}
&\left\langle \alpha', j', m' \mid [J_\pm, T_q^k] \mid \alpha, j, m \right\rangle \\
&= \hbar\sqrt{k(k+1) - q(q\pm 1)} \left\langle \alpha', j', m' \mid T_{q\pm1}^k \mid \alpha, j, m \right\rangle \\
\rightarrow \quad &\sqrt{(j'(j'+1) - m'(m'\mp 1)} \left\langle \alpha', j', m' \mp 1 \mid T_q^k \mid \alpha, j, m \right\rangle \\
&-\sqrt{j(j+1) - m(m\pm 1)} \left\langle \alpha', j', m' \mid T_k^k \mid \alpha, j, m \pm 1 \right\rangle \\
= \;& \hbar\sqrt{k(k+1) - q(q\pm 1)} \left\langle \alpha', j', m' \mid T_{q\pm1}^k \mid \alpha, j, m \right\rangle
\end{aligned}$$

We see that this is the same recursion relationship as for the Clebsch Gordon coefficients. Therefore it is reasonable to conclude that the matrix elements are all proportional to Clebsch Gordon coefficients which is precisely what the Wigner Eckart

Theorem says, namely

$$\left\langle \alpha', j', m' \mid T_q^k \mid \alpha, j, m \right\rangle = \langle jmkq \mid j', m', j, k \rangle \frac{\left\langle \alpha' j' \mid |T^k| \mid \alpha, j \right\rangle}{\sqrt{2j'+1}}$$

Operator for E2 transitions

An example of a tensor operator. The interaction of an electromagnetic field with a charged particle will correspond to a term in the Hamiltonian

$$H_{int} = \frac{e}{2mc}\mathbf{p} \cdot \mathbf{A}$$

If the fields are in the form of a plane wave then

$$\mathbf{A} = \epsilon A_0 e^{\mathbf{k} \cdot \mathbf{r}}$$

ϵ is the polarization vector. If $k = 2\pi\lambda$ is small or λ is large compared to the extent of the wave function (an atom), the we can expand in powers of k and

$$\mathbf{A} \sim \epsilon A_0 (1 + i\mathbf{k} \cdot \mathbf{r} - \frac{1}{2}(\mathbf{k} \cdot \mathbf{r})^2 + ...$$

and

$$H_{int} \sim \frac{eA_0}{2mc}\left(\epsilon \cdot \mathbf{p} + i(\epsilon \cdot \mathbf{p})(\mathbf{k} \cdot \mathbf{r}) + ...\right)$$

The second term is the E2 transition operator and it will have the form

$$E2_{ij} = p_i r_j$$

Implications

Matrix elements of a scalar operator T_0^0 are zero unless $m = m'$ and $j = j'$

$$\langle \alpha', j'm' \mid S \mid \alpha, j, m \rangle = \delta_{jj'} \delta_{mm'} \frac{\langle \alpha' j' \mid |S| \mid \alpha j \rangle}{\sqrt{2j+1}}$$

For a vector operator $\Delta m = \pm 1, 0$ and $\Delta j = \pm 1, 0$ and there is no $0 \to 0$ transition. In fact that transition is forbidden for any rank except 0. Must be a higher order, two photon transition.

Example of Wigner Eckart symmetry

Consider the dipole operator $\vec{\mathbf{r}}$. The matrix element

$$\begin{aligned}
\langle f \mid \vec{\mathbf{r}} \mid i\rangle &= \int R_f^*(r) Y_{l_f,m_f}^*(rY_{1,m}) R_i(r) Y_{l_i,m_i} d^3r \\
&= \int R_f^*(r) r R_i(r) r^2 dr \int Y_{l_f,m_f}^*(Y_{1,m}) Y_{l_i,m_i} d\Omega \\
&= \int R_f^*(r) r R_i(r) r^2 dr \langle l_i, m_i, 1, m \mid l_f, m_f\rangle \langle l_i, 0, 1, 0 \mid l_f, 0\rangle \sqrt{\frac{(2l_i+1)(2l+1)}{4\pi(2l_f+1)}} \\
&= \langle l_i, m_i, 1, m \mid l_f, m_f\rangle \langle l_i, 0, 1, 0 \mid l_f, 0\rangle \sqrt{\frac{(2l_i+1)(2l+1)}{4\pi(2l_f+1)}} \int R_f^*(r) r R_i(r) r^2 dr \\
r_m^1 &= \langle l_i, m_i, 1, m \mid l_f, m_f\rangle \frac{\langle f, l_f \mid |T^1| \mid i, l_i\rangle}{\sqrt{2l_f+1)}}
\end{aligned}$$

where

$$r_\pm = rY_{1,\pm 1}(\theta,\phi), \qquad r_- = rY_{1,0}(\theta,\phi)$$

Projection theorem

The theorem shows that an expectation value taken between states with the same j, that a rank one tensor can be written as $\lambda\mathbf{J}$ where λ is some constant that is independent of the z-component of angular momentum of the initial and final states.

$$\left\langle \alpha', jm \mid V_q^1 \mid \alpha, jm \right\rangle = \frac{\langle \alpha', jm \mid \mathbf{J}\cdot\mathbf{V} \mid \alpha, jm\rangle}{\hbar^2 j(j+1)} \langle jm' \mid J_q \mid jm\rangle .$$

First note that $\mathbf{J}\cdot\mathbf{V} = J_0V_0 - J_+V_- - J_-V_+$ Then

$$\begin{aligned}
\langle \alpha', j, m \mid \mathbf{J}\cdot\mathbf{V} \mid \alpha, j, m\rangle &= m\hbar \langle \alpha', jm \mid V_0 \mid \alpha, j, m\rangle \\
&\quad -\frac{\hbar}{2}\sqrt{j(j+1)-m(m-1)} \langle \alpha', jm-1 \mid V_- \mid \alpha, jm\rangle \\
&\quad -\frac{\hbar}{2}\sqrt{j(j+1)-m(m+1)} \langle \alpha', jm+1 \mid V_+ \mid \alpha, jm\rangle \\
&= c_{jm}\langle \alpha', j \,||\mathbf{V}||\, \alpha j\rangle
\end{aligned}$$

where the Wigner-Eckart theorem is used in the last step. But $\mathbf{J}\cdot\mathbf{V}$ is a scalar operator so its expectation value can have no m dependence. So $c_{jm} \to c_j$. We might have done the same exercise with $\mathbf{V} \to \mathbf{J}$ and then we would get

$$\left\langle \alpha, jm \mid \mathbf{J^2} \mid \alpha, jm \right\rangle = c_j \langle \alpha', j \,||\mathbf{J}||\, \alpha, j\rangle$$

Meanwhile we could have written from the WE theorem

$$\frac{\langle \alpha' jm' \mid V_q \mid \alpha, jm\rangle}{\langle \alpha, jm' \mid J_q \mid \alpha, jm\rangle} = \frac{\langle \alpha', j \, ||\mathbf{V}|| \, \alpha, j\rangle}{\langle \alpha, j \, ||\mathbf{J}|| \, \alpha j\rangle}$$

or

$$\frac{\langle \alpha' jm' \mid V_q \mid \alpha, jm\rangle}{\langle \alpha, jm' \mid J_q \mid \alpha, jm\rangle} = \frac{\langle \alpha', jm \, |\mathbf{J}\cdot\mathbf{V}| \, \alpha, jm\rangle}{\langle \alpha, jm \, |\mathbf{J}\cdot\mathbf{J}| \, \alpha jm\rangle}$$

which implies

$$\langle \alpha' jm' \mid V_q \mid \alpha, jm\rangle = \frac{\langle \alpha', jm \, |\mathbf{J}\cdot\mathbf{V}| \, \alpha, jm\rangle}{\hbar^2 j(j+1)} \langle \alpha, jm' \mid J_q \mid \alpha, jm\rangle$$

$\mathbf{J}\cdot\mathbf{V}$ is a scalar so its expectation value is independent of m. Therefore, the operator $\mathbf{V} = \lambda\mathbf{J}$ where λ is independent of m and m'.