P6574 HW #1

Due February 1, 2013

1. **Inelastic scattering**

   Show that even if the potential absorbs particles, we can describe it by

   $$S_l(k) = \eta_l(k) e^{2i\delta_l} \tag{1}$$

   where $\eta(< 1)$, is called the inelasticity factor and η and δ are real.

   (a) By considering the probability currents, show that

   $$\begin{aligned} \sigma_{inel} &= \frac{\pi}{k^2} \sum_{l=0}^{\infty} (2l+1)[1-\eta_l^2] \\ \sigma_{el} &= \frac{\pi}{k^2} \sum_{l=0}^{\infty} (2l+1)(1+\eta_l^2 - 2\eta_l \cos 2\delta_l) \end{aligned}$$

   and that

   $$\sigma_{tot} = \frac{4\pi}{k} \mathrm{Im} f(0) \tag{2}$$

   (b) Consider a "black disk" which absorbs everything for $r \leq r_0$ and is ineffective beyond. Idealize it by $\eta = 0$ for $l \leq kr_0; \eta = 1, \delta_0 = 0$ for $l > kr_0$. Show that $\sigma_{el} = \sigma_{inel} \approx \pi r_0^2$. Replace the sum by an integral and assume $kr_0 \gg 1$. Why is σ_{inel} always accompanied by σ_{el}?

2. **Optical theorem**

   (a) Show that the radial component of the current density due to interference between the incident and scattered waves is

   $$\lim_{r\to\infty} j_r^{int} \sim \left(\frac{\hbar k}{\mu}\right) \frac{1}{r} \mathrm{Im}\left[i e^{ikr(\cos\theta - 1)} f^*(\theta) \cos\theta + i e^{ikr(1-\cos\theta)} f(\theta) \right] \tag{3}$$

   (b) Argue that as long as $\theta \neq 0$, the average of j_r^{int} over any small solid angle is zero because $r \to \infty$. [Assume $f(\theta)$ is a smooth function.]

   (c) Integrate j_r^{int} over a tiny cone in the forward direction and show that (see hint)

   $$\int_{\text{forward cone}} j_r^{int} r^2 d\Omega = -\left(\frac{\hbar k}{\mu}\right) \frac{4\pi}{k} \mathrm{Im} f(0) \tag{4}$$

Thus, if we integrate the total current in the region behind the target, we find that the interference term (important only in the near-forward direction, behind the target) produces a depletion of particles, casting a "shadow". The total number of particles(per second) missing in the shadow region is given by the above expression for the integrated flux. Equating this loss to the product of the incident flux $\hbar k/\mu$ and the cross section σ, we regain the optical theorem. (Hint: Since θ is small, use a small angle approximation. In evaluating the upper limit in the θ integration, use the idea that the limit of a function that oscillates as its argument approaches infinity is equal to its average value.)

3. **Generalized optical theorem**

   The generalized optcal theorem reads

$$\begin{aligned} T_{ba} - T_{ab}^* &= -2\pi i \sum_i \delta(E - E_i) T_{bi} T_{ai}^* \\ &= -2\pi i \sum_f \delta(E - E_i) T_{fa} T_{fb}^* \quad \text{with} \quad E = E_a = E_b \end{aligned}$$

   Here the labels a and b specify initial and final states.

   (a) Show that the generalized optical theorem follows from the relation

$$T_{ba} = \langle \Phi_b \mid V \mid \Psi_a^R \rangle$$

   between the T matrix, the free state $\mid \Phi_b \rangle$, the scattering state $\mid \Psi_a^R \rangle$ and the interaction V, and the Lippmann-Schwinger equation.

   (b) Show that the optical theorem,

$$\mathrm{Im} T_{ii} = -\pi \sum_f \delta(E_i - E_f) |T_{fi}|^2,$$

   follows from the generalized optical theorem.

   (c) Use the generalized optical theorem to show that the scattering matrix,

$$S_{ba} = \delta_{ba} - 2\pi i T_{ba} \delta(E_b - E_a),$$

   is unitary.

4. **Wigner time delay**

   Measuring time in quantum mechanics is problematic, since there is no such thing as a time operator. Wigner has shown that the scattering phase shifts can be used to define the time a particle is delayed because of interactions with a scatterer. In order to understand Wigners idea in the simplest possible setting, we consider a one-dimensional example. We compare two potential profiles labeled V_1 and V_2. Both potentials vanish for $x > 0$,

$$V_1(x) = V_2(x) = 0 \quad \text{if } x > 0.$$

The potentials differ for $x < 0$. For the first potential, we take

$$V_1(x) \to \infty \ \text{ if } x < 0.$$

For the second potential, we take an arbitrary dependence on x, but with the condition that

$$V_2(x) \to \infty \quad \text{if} \quad x \to -\infty.$$

A schematic picture showing both potentials is shown in Fig. 1.

Obviously, a particle incident from the right will reflect off either potential. However, the time after which it returns will be longer in the case of potential V_2 than in the case of potential V_1. The time the particle reflecting off potential V_2 lags behind a particle reflecting off potential V_1 is referred to as the Wigner time delay τ_W.

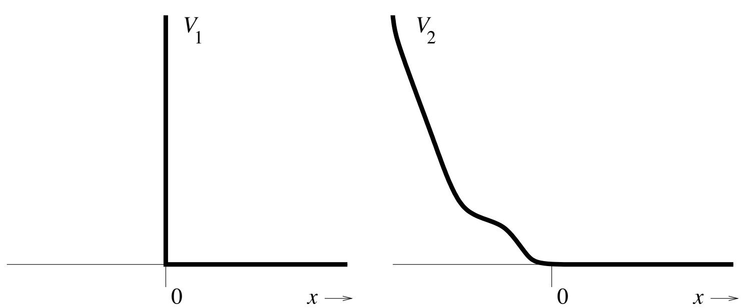

Figure 1: Schematic drawing of the potentials V_1 and V_2. The time a particle incident from the right and reflecting off potential V_2 lags behind a particle that reflects off potential V_1 is known as the "Wigner time delay".

(a) Model the incoming particle by a wavepacket, and find an expression for τ_W in terms of the derivative of the scattering phase shift δ to the particle's energy E.

(b) What is the maximal allowable energy uncertainty for the expression you derived under (a) to make sense? What does this imply for the relation between the minimal width of the wavepacket in the temporal domain and the delay time?

Note 1: Besides giving a quantitative relation between the energy-derivative of the scattering phase shift and the delay time, the result you find under (a) proves a very important qualitative result: The scattering matrix is a fast function of energy if the projectile spends a long time in the scattering region, and it is a slow function of energy if the projectile spends only a short time in the scattering region. This duality between time and energy is closely related to the Heisenberg uncertainty principle.

Note 2: Your analysis of part (b) shows that the Wigner delay time has little practical relevance as a delay time for particles that are well localized in the time domain. However, it plays an important role in solid state physics, where it is closely related to the density of states, a quantity that can be measured, e.g., using scanning probe techniques.