P6572 HW #10
Due November 11, 2011

1. **Sakurai, p. 284, problem 12**
   The Hamiltonian for a spin 1 system is given by

$$H = AS_z^2 + B(S_x^2 - S_y^2).$$

   Solve this problem *exactly* to find the normalized energy eigenstates and eigenvalues. (A spin-depenedent Hamiltonian of this kind actually appears in crystal physics.) Is this Hamiltonian invariant under time reversal? How do the normalized eigenstates you obtained transform under time reversal?

2. **Sakurai, p. 347, problem 7**
   A one-electron atom whose ground state is nondegenerate is placed in a uniform electric field in the $z-$ direction. Obtain an approximate expression for the induced electric dipole moment of the ground state by considering the expectation value of ez with respect to the perturbed state vector computed to first order. Show that the same expression can also be obtained from the energy shift $\Delta = -\alpha|\mathbf{E}|^2/2$ of the ground state computed to second order. ($Note$: α stands for the polarizability.) Ignore spin.

3. **Thomas-Reiche-Kuhn rule**

   (a) Prove the Thomas-Reiche-Kuhn sum rule

$$\sum_{n'}(E_{n'} - E_n)|\langle n' \mid X \mid n\rangle|^2 = \sum_{n'}(E_{n'} - E_n)\langle n \mid X \mid n'\rangle\langle n' \mid X \mid n\rangle = \frac{\hbar^2}{2m}$$

   where $\mid n\rangle$ and $\mid n'\rangle$ are eigenstates of $H = P^2/2m + V(X)$. (Hint: Eliminate the $E_{n'} - E_n$ factor in favor of H.)

   (b) Test the sum rule on the nth state of the oscillator

4. **Coupled oscillators**

   (a) A pair of identical one-dimensional oscilaltors is coupled together by a spring so that there is a force between then that is proportional to the difference of their displacements. Show that the hamiltonian is

$$\hat{H} = \frac{\hat{p}_1^2 + \hat{p}_2^2}{2m} + \frac{1}{2}k(x_1^2 + x_2^2) + \frac{1}{2}\alpha(x_1 - x_2)^2$$

which can be also written

$$\hat{H} = \frac{\hat{p}_1^2 + \hat{p}_2^2}{2m} + \frac{1}{2}(k+\alpha)(x_1^2 + x_2^2) - \alpha x_1 x_2$$

If the unperturbed oscillators are in states n' and n, find the first-order shift in the energy, proportional to α.

(b) Change the coordinates to a new pair

$$\xi_1 = \frac{1}{\sqrt{2}}(x_1 + x_2) \qquad \xi_2 = \frac{1}{\sqrt{2}}(x_1 - x_2)$$

and show that the energy levels can now be determined exactly. Compare the exact solution with that first-order shift.

[You may use $\langle n' \mid x \mid n \rangle = \sqrt{\hbar/2m\omega}(\sqrt{n+1}\delta_{n',n+1} + \sqrt{n}\delta_{n',n-1})$.]

5. **Spin 1/2 in magnetic field**
Consider a spin-1/2 particle with gyromagnetic ratio γ in a magnetic field $\mathbf{B} = B\mathbf{i} + B_0\mathbf{k}$. Treating B as a perturbation, calculate the first-and second-order shifts in energy and first-order shift in wave function for the ground state. Then compare the exact answers expanded to the corresponding orders.