Physics 6572 HW #2 Solutions

References below are to the following textbooks:

- Sakurai, Napolitano, *Modern Quantum Mechanics*, 2nd edition
- Goldstein, Poole, & Safko, *Classical Mechanics*, 3rd edition

Problem 1

Suppose that A and B are operators such that

$$\begin{aligned}[A,[A,B]] &= [B,[A,B]] \\ &= 0\end{aligned}$$

Part a)

We prove the identity:

$$[A^n, B] = n\,A^{n-1}\,[A,B]$$

for any nonnegative integer n. The proof is by induction. The cases $n = 0$ and $n = 1$ are trivial. Suppose that the identity holds for the exponent $n-1$. We have:

$$\begin{aligned}[A^n, B] &= A^n\,B - B\,A^n \\ &= A^{n-1}\,[A,B] + [A^{n-1}, B]\,A \\ &= A^{n-1}\,[A,B] + (n-1)\,A^{n-2}\,[A,B]\,A \\ &= n\,A^{n-1}\,[A,B]\end{aligned}$$

since $[A, [A, B]] = 0$. QED. Now suppose that $f(A)$ is a function of A defined by a Taylor series in non-negative powers of A, where the coefficients of the Taylor series are assumeed to commute with both A and B. It is easy to see that:

$$[f(A), B] = f'(A)\ [A,B]$$

where f' denotes the formal derivative of f applied to an operator argument A. $e^{x\,A} = \sum_n \frac{1}{n!}\,(x\,A)^n$ is such a function. Therefore,

$$\left[e^{x\,A}, B\right] = x\,e^{x\,A}\,[A,B]$$

Now define the operator

$$G(x) \equiv e^{x\,A}\,e^{x\,B}$$

By construction, $G(x)$ is invertible, with $G^{-1}(x)=e^{-xB}\,e^{-xA}$. We differentiate w.r.t. x:

$$\begin{aligned}\frac{\mathrm{d}G(x)}{\mathrm{d}x} &= \frac{\mathrm{d}\,e^{xA}}{\mathrm{d}x}\,e^{xB}+e^{xA}\,\frac{\mathrm{d}e^{xB}}{\mathrm{d}x}\\ &= A\,e^{xA}\,e^{xB}+e^{xA}\,e^{xB}\,B\\ &= (A+B)\,e^{xA}\,e^{xB}+[e^{xA},B]\,e^{xB}\\ &= (A+B+x\,[A,B])\,G(x)\end{aligned}$$

Informally, we integrate by separation of variables to obtain:

$$\log G(x)+k = x\,A+x\,B+\frac{1}{2}\,x^2\,[A,B]$$

where k is an integration constant. However, it's not clear that this is well defined for operators. Instead, define:

$$F(x) \equiv x\,A+x\,B+\frac{1}{2}\,x^2\,[A,B]$$

Thus, multiplying the above equation by $e^{-F(x)}$ on the left

$$e^{-F(x)}\,(G'(x)-F'(x)\,G(x)) = 0$$

Note that:

$$\begin{aligned}[F'(x),F(x)] &= \left[A+B+x\,[A,B],\,x\,A+x\,B+\frac{1}{2}\,x^2\,[A,B]\right]\\ &= \frac{1}{2}\,x^2\,[A+B,[A,B]]+x^2\,[[A,B],A+B]\\ &= 0\end{aligned}$$

since

$$\begin{aligned}[A,[A,B]] &= [B,[A,B]]\\ &= 0\end{aligned}$$

For any operator satisfying $[F'(x),F(x)]=0$, it's easy to show that:

$$\frac{\mathrm{d}}{\mathrm{d}x}\,e^{-F(x)} = -\,F'(x)\,e^{-F(x)}$$

Thus, the above differential equation becomes:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[e^{-F(x)}\,G(x)\right] = 0$$

This is easily integrated:

$$G(x) = e^{F(x)}\,G_0$$

where G_0 is an operator which does not depend on x. Checking the special case $x=0$, we find $G=1$ and $F=0$. Therefore $G_0=1$. Eliminating F and G in favor of A and B, we find:

$$e^{xA}\,e^{xB} \;=\; e^{xA+xB+\frac{1}{2}x^2[A,B]}$$

Setting $x=1$, this becomes:

$$e^{A}\,e^{B} \;=\; e^{A+B+\frac{1}{2}[A,B]}$$

Since $[A,B]$ commutes with A and B, this is equivalent to:

$$e^{A+B} \;=\; e^{A}\,e^{B}\,e^{-\frac{1}{2}[A,B]}$$

Part b)

Suppose that $\alpha,\beta\ll 1$. Expanding the exponentials in power series', we find:

$$\begin{aligned} e^{\alpha A}\,e^{\beta B} &= \left(1+\alpha A+\frac{1}{2}\alpha^2A^2+\ldots\right)\left(1+\beta B+\frac{1}{2}\beta^2B^2+\ldots\right) \\ &= 1+\alpha A+\beta B+\frac{1}{2}\alpha^2A+\alpha\beta AB+\frac{1}{2}\beta^2B+\mathcal{O}((\alpha|\beta)^3) \end{aligned}$$

where $\mathcal{O}((\alpha|\beta)^3)$ indicates that the omitted terms contain $\alpha^m\,\beta^n$ with $m+n\geqslant 3$. Take the log of both sides:

$$\begin{aligned} \log e^{\alpha A}\,e^{\beta B} &= \left(\alpha A+\beta B+\frac{1}{2}\alpha^2A+\alpha\beta AB+\frac{1}{2}\beta^2B+\mathcal{O}((\alpha|\beta)^3)\right) \\ &\quad -\frac{1}{2}\left(\alpha A+\beta B+\frac{1}{2}\alpha^2A+\alpha\beta AB+\frac{1}{2}\beta^2B+\mathcal{O}((\alpha|\beta)^3)\right)^2+\ldots \\ &= \alpha A+\beta B+\frac{1}{2}\alpha^2A+\alpha\beta AB+\frac{1}{2}\beta^2B-\frac{1}{2}(\alpha A+\beta B)^2+\mathcal{O}((\alpha|\beta)^3) \\ &= \alpha A+\beta B+\alpha\beta AB-\frac{1}{2}\alpha\beta\,\{A,B\}+\mathcal{O}((\alpha|\beta)^3) \\ &= \alpha A+\beta B+\frac{1}{2}\alpha\beta\,[A,B]+\mathcal{O}((\alpha|\beta)^3) \end{aligned}$$

Thus, exponentiating once more:

$$e^{\alpha A}\,e^{\beta B} \;=\; e^{\alpha A+\beta B+\frac{1}{2}\alpha\beta[A,B]+\mathcal{O}((\alpha|\beta)^3)}$$

Problem 2

Part a)

Recall from problem 1 that if A and B both commute with $[A,B]$ then

$$[f(A),B] \;=\; f'(A)\;[A,B]$$

where $f(A)$ can be defined using a Taylor series in A.

Setting $A = x$ and $B = p$, we find:

$$\begin{aligned}[f(x), p] &= f'(x)\,[x, p] \\ &= i\hbar f'(x)\end{aligned}$$

Similarly, setting $A = p$ and $B = x$, we find:

$$\begin{aligned}[g(p), x] &= g'(p)\,[p, x] \\ &= -i\hbar\, g'(p)\end{aligned}$$

In either case, f and g may depend on other operators which commute with both x and p. Thus, the formulae generalize immediately to multidimensional systems (i.e. systems with a set of x coordinates, x_i):

$$\begin{aligned}[x_i, G(\boldsymbol{p})] &= i\hbar\frac{\partial G}{\partial p_i} \\ [p_i, F(\boldsymbol{x})] &= -i\hbar\frac{\partial F}{\partial x_i}\end{aligned}$$

where we use the commutation relations:

$$\begin{aligned}[x_i, x_j] &= 0 \\ [p_i, p_j] &= 0 \\ [x_i, p_j] &= i\hbar\,\delta_{ij}\end{aligned}$$

Part b)

Using part a, we find:

$$\left[x, p^2\right] = 2i\hbar p$$

Thus,

$$\begin{aligned}\left[x^2, p^2\right] &= x^2\,p^2 - p^2\,x^2 \\ &= x\left[x, p^2\right] + \left[x, p^2\right]x \\ &= 2i\hbar\,\{x, p\} \\ &= 2i\hbar\,(2xp - [x, p]) \\ &= 4i\hbar\,xp + 2\hbar^2\end{aligned}$$

Part c)

Now consider the classical Poisson bracket:

$$\begin{aligned}\left[x^2, p^2\right]_{\text{classical}} &= \frac{\partial x^2}{\partial x}\frac{\partial p^2}{\partial p} - \frac{\partial x^2}{\partial p}\,\frac{\partial p^2}{\partial x} \\ &= 4x\,p\end{aligned}$$

Equation (1.6.47) in Sakurai states the general principle

$$[,]_{\text{classical}} \;\to\; \frac{1}{i\hbar}\,[,]$$

But

$$4i\hbar xp \;\neq\; 4i\hbar xp + 2\hbar^2$$

This is what's known as an *ordering ambiguity*. In the classical theory, we are free to rewrite:

$$\begin{aligned}\left[x^2, p^2\right]_{\text{classical}} &= 4xp \\ &= 2\,(xp + px) \\ &= 4px\end{aligned}$$

since x and p are just numbers. However, upon quantization, only the middle line reproduces the correct quantum mechanical result, namely

$$\left[x^2, p^2\right] \;=\; 2i\hbar\,\{x, p\}$$

Thus, given a classical system, there may be more than one way to quantize it consistent with the correspondence principle, depending on what orderings we choose upon promoting the Poisson brackets to commutators.[1] As Sakurai states on p. 84, "classical mechanics can be derived from quantum mechanics, but the opposite is not true."

Problem 3

Define the translation operator:

$$\mathcal{T}(\boldsymbol{l}) \;\equiv\; \exp\left[\frac{\boldsymbol{p}\cdot\boldsymbol{l}}{i\hbar}\right]$$

Part a)

Using the result of part (a) of the previous problem, we find:

$$\begin{aligned}[x_i, \mathcal{T}(\boldsymbol{l})] &= i\hbar\left(\frac{1}{i\hbar}\,l_i\right)\exp\left[\frac{\boldsymbol{p}\cdot\boldsymbol{l}}{i\hbar}\right] \\ &= l_i\,\mathcal{T}(\boldsymbol{l})\end{aligned}$$

Part b)

Consider a state $|\psi\rangle$. Now translate $|\psi\rangle$ using $\mathcal{T}(\boldsymbol{l})$:

$$|\psi'\rangle \;=\; \mathcal{T}(\boldsymbol{l})\,|\psi\rangle$$

1. Since the $[x^2, p^2]$ commutator can be derived from the $[x, p]$ commutator, which has no ordering ambiguities, this does not happen in this simple case. However, it does occur for certain (more complicated) systems.

Thus,

$$\begin{aligned}
\langle \boldsymbol{x} \rangle' &= \langle \psi' | \boldsymbol{x} | \psi' \rangle \\
&= \big\langle \psi | \mathcal{T}^{-1}(\boldsymbol{l})\, \boldsymbol{x}\, \mathcal{T}(\boldsymbol{l}) | \psi \big\rangle \\
&= \big\langle \psi | \mathcal{T}^{-1}(\boldsymbol{l})\, \mathcal{T}(\boldsymbol{l})\, \boldsymbol{x} + \mathcal{T}^{-1}(\boldsymbol{l})\, [\boldsymbol{x}, \mathcal{T}(l)]\, | \psi \big\rangle \\
&= \langle \psi | \boldsymbol{x} | \psi \rangle + \big\langle \psi | \mathcal{T}^{-1}(\boldsymbol{l})\, \boldsymbol{l}\, \mathcal{T}(\boldsymbol{l}) | \psi \big\rangle \\
&= \langle \boldsymbol{x} \rangle + \boldsymbol{l}
\end{aligned}$$

since $\boldsymbol{l}$ is an ordinary vector, and therefore commutes with $\mathcal{T}(\boldsymbol{l})$, and $\langle \psi | \psi \rangle = 1$. This is just what one would expect the translation operator to do.

Problem 4

Consider the transformation:

$$\begin{aligned}
Q &= \log\left(\frac{1}{q}\sin p\right) \\
P &= q \cot p
\end{aligned}$$

As explanined in the assignment, the transformation is canonical iff the Poisson bracket $[Q,\ P]_{q,p} = 1$ (Goldstein section 9.5). We find:

$$\begin{aligned}
[Q, P]_{q,p} &= \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} \\
&= \left(-\frac{1}{q}\right)\big(-q \csc^2 p\big) - \left(\frac{1}{\sin p}\cos p\right)(\cot p) \\
&= \csc^2 p - \cot^2 p \\
&= 1
\end{aligned}$$

so the transformation is indeed canonical.

Another way to show that this transformation is canonical is to obtain the generating function (Goldstein section 9.1). We solve for q in terms of Q and p using the first equation:

$$q = e^{-Q}\sin p$$

Putting this into the second equation, we find:

$$P = e^{-Q}\cos p$$

Referring to Goldstein table 9.1, we look for a generating function of the form:

$$F = F_3(p, Q) + p\, q$$

where F_3 must satisfy:

$$\begin{aligned}
q &= -\frac{\partial F_3}{\partial p} \\
P &= -\frac{\partial F_3}{\partial Q}
\end{aligned}$$

The answer is easy to guess:

$$F_3 \;=\; e^{-Q}\cos p$$

Thus, the transformation is canonical.

Problem 5

This problem is easier if we start with part b:

Part b)

Consider the operator

$$\mathcal{K}(\Xi) \;=\; \exp\left[\frac{i x \Xi}{\hbar}\right]$$

The analogous operator involving the momentum generated translations (shifts in position). Thus, we guess that this operator generates boosts (shifts in momentum).

To check this, we repeat the computation of problem 3:

$$\begin{aligned}[p, \mathcal{K}(\Xi)] \;&=\; -\,i\hbar\left(\frac{i\,\Xi}{\hbar}\right)\mathcal{K}(\Xi) \\ &=\; \Xi\,\mathcal{K}(\Xi)\end{aligned}$$

Thus, for a state $|\psi\rangle$, the boosted state is given by:

$$|\psi'\rangle \;=\; \mathcal{K}(\Xi)\,|\psi\rangle$$

We find:

$$\begin{aligned}\langle p\rangle' \;&=\; \langle\psi'|p|\psi'\rangle \\ &=\; \left\langle\psi|\mathcal{K}^{-1}(\Xi)\;p\,\mathcal{K}(\Xi)|\psi\right\rangle \\ &=\; \left\langle\psi|\mathcal{K}^{-1}(\Xi)\;\mathcal{K}(\Xi)\,p + \mathcal{K}^{-1}(\Xi)\;[p, \mathcal{K}(\Xi)]\,|\psi\right\rangle \\ &=\; \langle\psi|p|\psi\rangle + \left\langle\psi|\mathcal{K}^{-1}(\Xi)\,\Xi\,\mathcal{K}(\Xi)|\psi\right\rangle \\ &=\; \langle p\rangle + \Xi\end{aligned}$$

Thus, $\mathcal{K}$ generates boosts as expected. For a momentum eigenstate:

$$\begin{aligned}p\,|q'\rangle \;&=\; p\,\mathcal{K}(\Delta\,q)\,|q\rangle \\ &=\; [p, \mathcal{K}(\Delta\,q)]\,|q\rangle + \mathcal{K}(\Delta\,q)\;p\,|q\rangle \\ &=\; (q + \Delta\,q)\,\mathcal{K}(\Delta\,q)\,|q\rangle\end{aligned}$$

Thus,

$$\mathcal{K}(\Delta\,q)\,|q\rangle \;=\; |q + \Delta\,q\rangle$$

Part a)

i)

We loosely follow the logic of Sakurai eqns 1.7.15 – 1.7.17. x is the "generator" of boosts:

$$x = -i\hbar \lim_{\Delta p \to 0} \frac{1}{\Delta p} \left(\mathcal{K}(\Delta p) - \mathcal{K}(0)\right)$$

where $\mathcal{K}(0) = \text{id}$. Consider x acting on a state $|\alpha\rangle$:

$$x\,|\alpha\rangle = -i\hbar \lim_{\Delta p \to 0} \frac{1}{\Delta p} \left(\mathcal{K}(\Delta p)\,|\alpha\rangle - |\alpha\rangle\right)$$

Contract with $|p\rangle$ to obtain:

$$\begin{aligned}\langle p|x|\alpha\rangle &= -i\hbar \lim_{\Delta p \to 0} \frac{\langle p - \Delta p|\alpha\rangle - \langle p|\alpha\rangle}{\Delta p} \\ &= i\hbar \frac{\partial}{\partial p} \langle p|\alpha\rangle\end{aligned}$$

where $\langle p|\mathcal{K}(\Delta p) = \langle p - \Delta p|$.

ii)

Now consider the the matrix elements of x:

$$\begin{aligned}\langle\beta|x|\alpha\rangle &= \int \mathrm{d}p'\,\langle\beta|p'\rangle\langle p'|x|\alpha\rangle \\ &= \int \mathrm{d}p'\,\langle\beta|p'\rangle i\hbar \frac{\partial}{\partial p'} \langle p'|\alpha\rangle \\ &= \int \mathrm{d}p'\,\phi^{\star}_{\beta}(p') \left(i\hbar \frac{\partial}{\partial p'}\right) \phi_{\alpha}(p')\end{aligned}$$

where

$$\begin{aligned}\phi_{\alpha}(p') &\equiv \langle p'|\alpha\rangle \\ \phi_{\beta}(p') &\equiv \langle p'|\beta\rangle\end{aligned}$$

Problem 6

A classical harmonic oscillator has the Lagrangian:

$$L = \frac{1}{2} m\,\dot{x}^2 - \frac{1}{2} m\,\omega^2\,x^2$$

The equation of motion is just:

$$\ddot{x} = -\omega^2\,x$$

The general solution takes the form:

$$x(t) = A\cos\omega t + B\sin\omega t$$

The total energy is:

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega^2 x^2$$

Putting in the above solution:

$$\begin{aligned} E &= \frac{1}{2}m\,\omega^2(-A\sin\omega t + B\cos\omega t)^2 + \frac{1}{2}m\,\omega^2(A\cos\omega t + B\sin\omega t)^2 \\ &= \frac{1}{2}m\,\omega^2\left(A^2 + B^2\right) \end{aligned}$$

Suppose that $x(0) = x_1$ and $x(T) = x_2$ for some time T. Thus,

$$\begin{aligned} A &= x_1 \\ A\cos\omega T + B\sin\omega T &= x_2 \end{aligned}$$

Solving for B in terms of x_1, x_2 and T, we find:

$$B = \frac{1}{\sin\omega T}[x_2 - x_1\cos\omega T]$$

Thus,

$$\begin{aligned} E &= \frac{1}{2}m\,\omega^2\left(x_1^2 + \frac{1}{\sin^2\omega T}[x_2 - x_1\cos\omega T]^2\right) \\ &= \frac{m\,\omega^2}{2\sin^2\omega T}\left(x_1^2 + x_2^2 - 2x_1x_2\cos\omega T\right) \end{aligned}$$

To compute the action, we first evaluate the Lagrangian:

$$\begin{aligned} L &= \frac{1}{2}m\,\omega^2(-A\sin\omega t + B\cos\omega t)^2 - \frac{1}{2}m\,\omega^2(A\cos\omega t + B\sin\omega t)^2 \\ &= \frac{1}{2}m\,\omega^2\left(B^2 - A^2\right)\left(\cos^2\omega t - \sin^2\omega t\right) - 2m\,\omega^2 AB\sin\omega t\cos\omega t \\ &= \frac{1}{2}m\,\omega^2\left[\left(B^2 - A^2\right)\cos 2\omega t - 2AB\sin 2\omega t\right] \end{aligned}$$

Integrating, we obtain:

$$\begin{aligned} S_{\mathrm{cl}} &= \int_0^T L\,\mathrm{d}t \\ &= \frac{1}{4}m\,\omega\left[\left(B^2 - A^2\right)\sin 2\omega t + 2AB\cos 2\omega t\right]_{t=0}^{t=T} \\ &= \frac{1}{4}m\,\omega\left[\left(B^2 - A^2\right)\sin 2\omega T + 2AB\left(\cos 2\omega T - 1\right)\right] \\ &= \frac{1}{2}m\,\omega\sin\omega T\left[\left(B^2 - A^2\right)\cos\omega T - 2AB\sin\omega T\right] \end{aligned}$$

Putting in the formulae for A and B in terms of x_1, x_2 and T, we find:

$$\begin{aligned}\left(B^2 - A^2\right)\cos\omega T - 2AB\sin\omega T &= \frac{1}{\sin^2\omega T}\,[x_2 - x_1\cos\omega T]^2\cos\omega T - x_1^2\cos\omega T - 2x_1\,[x_2 - x_1\cos\omega T] \\ &= \frac{1}{\sin^2\omega T}\left[\cos\omega T\,\left[x_1^2 + x_2^2\right] - 2\,x_1\,x_2\right]\end{aligned}$$

Thus,

$$S_{\mathrm{cl}}(x_1, x_2, T) = \frac{m\,\omega}{2\sin\omega T}\left[\left(x_1^2 + x_2^2\right)\cos\omega T - 2\,x_1\,x_2\right]$$

Suppose we vary the time taken to travel between x_1 and x_2 while fixing the positions x_1 and x_2. Thus,

$$\begin{aligned}\frac{\partial S_{\mathrm{cl}}}{\partial t} &\equiv \left.\frac{\partial S_{\mathrm{cl}}}{\partial T}\right|_{x_1, x_2} \\ &= \frac{1}{2}\,m\,\omega^2\,\frac{-\left[\left(x_1^2 + x_2^2\right)\sin\omega T\right]\sin\omega T - \left[\left(x_1^2 + x_2^2\right)\cos\omega T - 2\,x_1\,x_2\right]\cos\omega T}{\sin^2\omega T} \\ &= -\frac{m\omega^2}{2\sin^2\omega T}\left[x_1^2 + x_2^2 - 2\,x_1\,x_2\,\cos\omega T\right] \\ &= -E\end{aligned}$$

This is the Hamilton-Jacobi equation for the 1D simple harmonic oscillator (see Goldstein section 10.1).

Problem 7

Suppose that A and B are commuting Hermitean matrices. We choose a basis $|a\rangle$ of eigenvectors of A with eigenvalues λ_a. We find:

$$A(B\,|a\rangle) = B(A\,|a\rangle) = \lambda_a\,B\,|a\rangle$$

Thus $B\,|a\rangle$ is also an eigenvector of A for any eigenvector $|a\rangle$ of A.

There are two possibilities:

1. If the eigenvalue λ_a has no degeneracy, then we must have $B\,|a\rangle \propto |a\rangle$, so that $|a\rangle$ is also an eigenvector of B.

2. If, on the other hand, there exists a degenerate subspace $\{|\alpha\rangle\}$, all with eigenvalue λ_a under A, then B may have a nontrivial action within this subspace. Since B restricted to this subspace is still Hermitean, we can diagonalize it within this subspace by a change of basis. As $A = \lambda_a\,\mathbf{1}$ when restricted to this subspace, it remains diagonal under the change of basis.

Applying either 1 or 2 to each degenerate subspace, we obtain a basis $|a'\rangle$ of eigenvectors, which diagonalizes both A and B:

$$\begin{aligned}A &= \sum_{a'} \lambda_{a'}^{(A)}\,|a'\rangle\langle a'| \\ B &= \sum_{a'} \lambda_{a'}^{(B)}\,|a'\rangle\langle a'|\end{aligned}$$

This procedure can be iterated for any set of $n \geqslant 2$ commuting Hermitean operators (observables).

Conversely, suppose that A and B can be simultaneous diagonalized in some basis $|a\rangle$:

$$
\begin{aligned}
A &= \sum_a \lambda_a^{(A)} |a\rangle\langle a| \\
B &= \sum_a \lambda_a^{(B)} |a\rangle\langle a|
\end{aligned}
$$

Thus,

$$
AB = \sum_{a,b} \lambda_a^{(A)} \lambda_b^{(B)} |a\rangle\langle a|b\rangle\langle b| = \sum_a \lambda_a^{(A)} \lambda_a^{(B)} |a\rangle\langle a| = BA
$$

so A and B commute.