1. Consider a 1-dimensional hamiltonian $H = \frac{p^2}{2m} + V(x)$, $[x, p] = i\hbar$ and $H | \phi_n \rangle = E_n | \phi_n \rangle$

(a) Show that $\langle \phi_n | p | \phi_{n'} \rangle = \alpha \langle \phi_n | x | \phi_{n'} \rangle$. Determine $\alpha$.

(b) Derive the "sum rule"

$$\sum_{n'} (E_n - E_{n'})^2 | \langle \phi_n | x | \phi_{n'} \rangle |^2 = \frac{\hbar^2}{m^2} \langle \phi_n | p^2 | \phi_n \rangle$$

2. Canonical transformation and simple harmonic motion

The hamiltonian is

$$H = \frac{p^2}{2m} + \frac{1}{2} kq^2$$

Consider the generating function

$$F_1(q, \bar{q}) = \frac{m}{2} \omega q^2 \cot \bar{q}$$

where $\omega = \sqrt{k/m}$ and the equations relating old and new coordinates

$$\frac{\partial F_1}{\partial q} = p, \quad \frac{\partial F_1}{\partial \bar{q}} = -\bar{p}, \quad K(\bar{q}, \bar{p}) = H(p, q) + \frac{\partial F_1}{\partial t}$$

(a) Determine $q$ and $p$ in terms of $\bar{q}$ and $\bar{p}$ and write $K(\bar{q}, \bar{p}) = H(q, p)$.

(b) Use Hamilton’s equations to write the equations of motion in the transformed coordinate system.

(c) Integrate the equations of motion in the $\bar{q}, \bar{p}$ system. To what physical quantities do $\bar{q}$ and $\bar{p}$ correspond?

(d) Transform back to the $q, p$ system to determine $q(t)$ and $p(t)$

3. Expectation value

(a) Show that for a real wave function $\psi(x)$, the expectation value of momentum $\langle p \rangle = 0$. (Hint: Show that the probabilities for the momenta $\pm p$ are equal.) Generalize this result to the case $\psi = c \psi_r$, where $\psi_r$ is real and $c$ an arbitrary (real or complex) constant. (Recall that $| \psi \rangle$ and $\alpha | \psi \rangle$ are physically equivalent.)

(b) Show that if $\psi(x)$ has expectation value $\langle p \rangle$, $e^{ip_{0}/\hbar} \psi(x)$ has expectation value $\langle p \rangle + p_0$.

4. Harmonic oscillator

For the harmonic oscillator show that

$$\langle a(t) \rangle = e^{-i\omega t} \langle a(0) \rangle \quad \text{and that} \quad \langle a^\dagger(t) \rangle = e^{i\omega t} \langle a^\dagger(0) \rangle$$

5. Sakurai, p. 64, problem 18
(a) The simplest way to derive the Schwartz inequality goes as follows. First observe
\[
\langle \alpha | + \lambda^* \langle \beta | \cdot (| \alpha \rangle + \lambda | \beta \rangle) \geq 0
\]
for any complex number \( \lambda \); then choose \( \lambda \) in such a way that the preceding inequality reduces to the Schwartz inequality.

(b) Show that the equality sign in the generalized uncertainty relation holds if the state in question satisfies
\[
\Delta A | \alpha \rangle = \lambda \Delta B | \alpha \rangle
\]
with \( \lambda \) purely imaginary.

(c) Explicit calculations using the usual rules of wave mechanics show that the wave function for a Gaussian wave packet given by
\[
\langle x' | \alpha \rangle = \left( \frac{2\pi d^2}{\hbar} \right)^{-\frac{1}{4}} \exp \left[ \frac{i(p)x'}{\hbar} - \frac{(x' - \langle x \rangle)^2}{4d^2} \right]
\]
satisfies the minimum uncertainty relation
\[
\sqrt{\langle (\Delta x)^2 \rangle} \sqrt{\langle (\Delta p)^2 \rangle} = \frac{\hbar}{2}.
\]
Prove that the requirement
\[
\langle x' | \Delta x | \alpha \rangle = (\text{imaginary number}) \langle x' | \Delta p | \alpha \rangle
\]
is indeed satisfied for such a Gaussian wave packet, in agreement with (b).

6. **Sakurai, p. 143, problem 1**

Consider the spin-precession problem discussed in the text. It can also be solved in the Heisenberg picture. Using the Hamiltonian
\[
H = -\left( \frac{eB}{mc} \right) S_z = \omega S_z,
\]
write the Heisenberg equations of motion for the time-dependent operators \( S_x(t), S_y(t), \) and \( S_z(t) \). Solve them to obtain \( S_{x,y,z} \) as functions of time.

7. **Hamilton-Jacobi equation**

We can also analyze the harmonic oscillator using Hamilton-Jacobi theory. The Hamilton-Jacobi equation in one-dimension is
\[
\frac{1}{2m} \left( \frac{\partial S}{\partial q} \right)^2 + V(q) + \frac{\partial S}{\partial t} = 0
\]
\( S(q, \bar{p}) \) is the generator of the canonical transformation from \( H(q, p) \) to \( K(\bar{q}, \bar{p}) = 0 \). The old and new coordinates are related by
\[
\frac{\partial S}{\partial q} = p, \quad \frac{\partial S}{\partial \bar{p}} = \bar{q}
\]
Since \( K(\bar{q}, \bar{p}) = 0, \dot{\bar{p}} = -\frac{\partial K}{\partial \bar{q}} = 0 \) and \( \bar{p} \) is a constant of the motion.
Solve the Hamilton Jacobi equation for the harmonic oscillator Hamiltonian
\[
H(p, q) = \frac{p^2}{2m} + \frac{1}{2} kq^2
\]
Note that the generating function depends on \( q, t \) and a constant of the motion \( \bar{p} = \alpha \).
(a) Since the explicit dependence of $S$ on $t$ is involved only in the last term, a solution can be found of the form

\[ S(q, \alpha, t) = W(q, \alpha) - \alpha t \]

Find $W(q, \alpha)$ and $S(q, \alpha, t)$.

(b) Differentiate $S$ to determine $\ddot{q}$, (Hint: $\frac{\partial S}{\partial \alpha} = \ddot{q}$)

(c) Solve for $q$ in terms of $t$ and the integration constants.

(d) Suppose at time $t = 0$ that particle is initially stationary, $p(0) = 0$, but is displaced from equilibrium by $q(0)$. Determine $\ddot{p} = \alpha$, and $\ddot{q}$ in terms of the initial conditions.