P6572 HW #3
Due September 16, 2011

1. Consider a 1-dimensional hamiltonian $H = \frac{p^2}{2m} + V(x),\ [x,p] = i\hbar$ and $H|\,\phi_n\rangle = E_n|\,\phi_n\rangle$

   (a) Show that $\langle \phi_n \mid p \mid \phi_{n'}\rangle = \alpha \langle \phi_n \mid x \mid \phi_{n'}\rangle$.
   Determine α.

   (b) Derive the "sum rule"

$$\sum_{n'} (E_n - E_{n'})^2 |\langle \phi_n \mid x \mid \phi_{n'}\rangle|^2 = \frac{\hbar^2}{m^2}\left\langle \phi_n \mid p^2 \mid \phi_n \right\rangle$$

2. **Canonical transformation and simple harmonic motion**
   The hamiltonian is

$$H = \frac{p^2}{2m} + \frac{1}{2}kq^2$$

   Consider the generating function

$$F_1(q,\bar{q}) = \frac{m}{2}\omega q^2 \cot \bar{q}$$

   where $\omega = \sqrt{k/m}$ and the equations relating old and new coordinates

$$\frac{\partial F_1}{\partial q} = p, \qquad \frac{\partial F_1}{\partial \bar{q}} = -\bar{p}, \quad K(\bar{q},\bar{p}) = H(p,q) + \frac{\partial F_1}{\partial t}$$

   (a) Determine q and p in terms of $\bar{q}$ and $\bar{p}$ and write $K(\bar{q},\bar{p}) = H(q,p)$.

   (b) Use Hamilton's equations to write the equations of motion in the transformed coordinate system.

   (c) Integrate the equations of motion in the $\bar{q},\bar{p}$ system. To what physical quantities do $\bar{q}$ and $\bar{p}$ correspond?

   (d) Transform back to the q,p system to determine $q(t)$ and $p(t)$

3. **Expectation value**

   (a) Show that for a real wave function $\psi(x)$, the expectation value of momentum $\langle p\rangle = 0$. (Hint: Show that the probabilities for the momenta $\pm p$ are equal.) Generalize this result to the case $\psi = c\psi_r$, where ψ_r, is real and c an arbitrary (real or complex) constant. (Recall that $|\,\psi\rangle$ and $\alpha|\,\psi\rangle$ are physically equivalent.)

   (b) Show that if $\psi(x)$ has expectation value $\langle p\rangle$, $e^{ip_0x/\hbar}\psi(x)$ has expectation value $\langle p\rangle + p_0$.

4. **Harmonic oscillator**
   For the harmonic oscillator show that

$$\langle a(t)\rangle = e^{-i\omega t}\langle a(0)\rangle \quad \text{and that} \quad \langle a^\dagger(t)\rangle = e^{i\omega t}\langle a^\dagger(0)\rangle.$$

5. **Sakurai, p. 64, problem 18**

(a) The simplest way to derive the Schwartz inequality goes as follows. First observe

$$(\langle\alpha \mid + \lambda^*\langle\beta \mid) \cdot (\mid \alpha\rangle + \lambda\mid \beta\rangle) \geq 0$$

for any complex number λ; then choose λ in such a way that the preceding inequality reduces to the Schwartz inequality.

(b) Show that the equality sign in the generalized uncertainty relation holds if the state in question satisfies

$$\Delta A\mid \alpha\rangle = \lambda\Delta B\mid \alpha\rangle$$

with λ purely *imaginary*.

(c) Explicit calculations using the usual rules of wave mechanics show that the wave function for a Gauussian wave packet given by

$$\langle x' \mid \alpha\rangle = (2\pi d^2)^{-\frac{1}{4}} \exp\left[\frac{i\langle p\rangle x'}{\hbar} - \frac{(x' - \langle x\rangle)^2}{4d^2}\right]$$

satisfies the minimum uncertainty relation

$$\sqrt{\langle(\Delta x)^2\rangle}\sqrt{\langle(\Delta p)^2\rangle} = \frac{\hbar}{2}.$$

Prove that the requirement

$$\langle x' \mid \Delta x \mid \alpha\rangle = (\text{imaginary number})\, \langle x' \mid \Delta p \mid \alpha\rangle$$

is indeed satisfied for such a Gaussian wave packet, in agreement with (b).

6. **Sakurai, p. 143, problem 1**
Consider the spin-precession problem discussed in the text. It can also be solved in the Heisenberg picture. Using the Hamiltonian

$$H = -\left(\frac{eB}{mc}\right) S_z = \omega S_z,$$

write the Heisenberg equations of motion for the time-dependent operators $S_x(t), S_y(t)$, and $S_z(t)$. Solve them to obtain $S_{x,y,z}$ as functions of time.

7. **Hamilton-Jacobi equation**
We can also analyze the harmonic oscillator using Hamilton-Jacobi theory. The Hamilton-Jacobi equation in one-dimension is

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + V(q) + \frac{\partial S}{\partial t} = 0$$

$S(q, \bar{p})$ is the generator of the canonical transformation from $H(q, p)$ to $K(\bar{q}, \bar{p}) = 0$. The old and new coordinates are related by

$$\frac{\partial S}{\partial q} = p, \qquad \frac{\partial S}{\partial \bar{p}} = \bar{q}$$

Since $K(\bar{q}, \bar{p}) = 0$, $\dot{\bar{p}} = -\frac{\partial K}{\partial \bar{q}} = 0$ and $\bar{p}$ is a constant of the motion.
Solve the Hamilton Jacobi equation for the harmonic oscillator Hamiltonian

$$H(p, q) = \frac{p^2}{2m} + \frac{1}{2}kq^2$$

Note that the generating function depends on q, t and a constant of the motion $\bar{p} = \alpha$.

(a) Since the explicit dependence of S on t is involved only in the last term, a solution can be found of the form

$$S(q, \alpha, t) = W(q, \alpha) - \alpha t$$

Find $W(q, \alpha)$ and $S(q, \alpha, t)$.

(b) Differentiate S to determine $\bar{q}$, (Hint: $\frac{\partial S}{\partial \alpha} = \bar{q}$)

(c) Solve for q in terms of t and the integration constants.

(d) Suppose at time $t = 0$ that particle is initially stationary, $p(0) = 0$, but is displaced from equilibrium by $q(0)$. Determine $\bar{p} = \alpha$, and $\bar{q}$ in terms of the initial conditions.