P6572 HW #4

Due September 23, 2011

1. **SHO propagator**

   We know that given the eigenfunctions and the eigenvalues we can construct the propagator:

$$K(x,t;x',t') = \sum_n \psi_n(x)\psi_n^*(x')e^{-iE_n(t-t')/\hbar} \tag{1}$$

   Consider the reverse process (since the path integral approach gives K directly), for the case of the harmonic oscillator where

$$K(x,t;x',0) = \sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega t}}\exp\left\{\frac{im\omega}{2\hbar\sin\omega t}[(x^2+x'^2)\cos\omega t - 2xx']\right\}$$

   (a) Set $x = x' = t' = 0$. By expanding both sides of Equation 1 you should find that $E = \hbar\omega/2, 5\hbar\omega/2, 9\hbar\omega/2, ...,$etc. What happened to the levels in between?

   (b) Now consider the extraction of the eigenfunctions. Let $x = x'$ and $t' = 0$. Find $E_0, E_1, |\psi_0(x)|^2$, and $|\psi_1(x)|^2$ by expanding in powers of $a = \exp(i\omega t)$.

2. **Sakurai, p. 149, problem 30**

   The propagator in position space is (2.5.26), namely,

$$K(\mathbf{x}'',t,\mathbf{x}',t_0) = \sum_{a'}\left\langle \mathbf{x}'' \mid \exp(\frac{-iHt}{\hbar}) \mid a'\right\rangle\left\langle a' \mid \exp(\frac{iHt_0}{\hbar}) \mid \mathbf{x}'\right\rangle = \langle \mathbf{x}'',t \mid \mathbf{x}',t_0\rangle$$

   The analogous propagator in momentum space is given by $\langle \mathbf{p}'',t \mid \mathbf{p}',t_0\rangle$. Derive an explicit expression for $\langle \mathbf{p}'',t \mid \mathbf{p}',t_0\rangle$ for the free particle case.

3. **Green's function**

   Derive the Green's function $G(x,x',E)$ for a free particle in one dimension, where

$$\left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - E\right)G(x,x',E) = \delta(x-x')$$

4. **Sakurai, p. 149, Problem 26**

   Consider a particle moving in one dimension under the influence of a potential $V(x)$. Suppose its wave function can be written as $\exp[iS(x,t)/\hbar]$. Prove that $S(x,t)$ satisfies the classical Hamilton-Jacobi equation to the extent that $\hbar$ can be regarded as small in some sense. Show how one may obtain the correct wave function for a plane wave by starting with the solution of the classical Hamilton-Jacobi equation with $V(x)$ set equal to zero. Why do we get the exact wave function in this particular case?

5. **Coherent state**

A coherent state of a one-dimensional simple harmonic oscillator is defined to be an eigenstate of the (non-Hermitian) annihilation operator a:

$$a|\,\lambda\rangle = \lambda|\,\lambda\rangle,$$

where λ is, in general, a complex number.

(a) Prove that

$$|\,\lambda\rangle = e^{-|\lambda|^2/2} e^{\lambda a^\dagger}|\,0\rangle$$

is a normalized coherent state.

(b) Prove the minimum uncertainty relation for such a state.

(c) Write $|\,\lambda\rangle$ as

$$|\,\lambda\rangle = \sum_{n=0}^{\infty} f(n)|\,n\rangle.$$

Show that the distribution of $|f(n)|^2$ with respect to n is of the Poisson form. Find the most probable value of n, hence of E.

(d) Show that a coherent state can also be obtained by applying the translation (finite-displacement) operator $e^{-ipl/\hbar}$ (where p is the momentum operator, and l is the displacement distance) to the ground state.

(e) Show that the wave function of the coherent state is

$$\psi_\lambda(x) = \langle x \mid \lambda\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\lambda^2/2} e^{-(m\omega/2\hbar)x^2} e^{\sqrt{(2m\omega/\hbar)}\lambda x}$$

Start by using $a|\,\lambda\rangle = \lambda|\,\lambda\rangle$ in the coordinate representation. Fix the normalization by demanding that $\langle\lambda' \mid \lambda\rangle = e^{\lambda^*\lambda}$. (Hint: The identity $e^A e^B = e^B e^A e^{[A,B]}$ which is true if $[A,B]$ commutes with A and B might be useful.) Show that $\psi_\lambda(x,t)$ evolves with the time like classical coordinates given that $|\,\lambda\rangle \to |\,\lambda e^{-i\omega t}\rangle$.

6. **Sakurai, p. 150, problem 35**

Consider the Hamiltonian of a spinless particle of charge q. In the presence of a static magnetic field, the interaction terms can be generated by

$$\mathbf{p}_{\text{operator}} \to \mathbf{p}_{\text{operator}} - \frac{q\mathbf{A}}{c},$$

where $\mathbf{A}$ is the appropriate vector potential. Suppose, for simplicity, that the magnetic field $\mathbf{B}$ is uniform in the positive z-direction. Prove that the above prescription indeed leads to the correct expression for the interaction of the orbital magnetic moment $(q/2mc)\mathbf{L}$ with the magnetic field $\mathbf{B}$. Show that there is also an extra term proportional to $B^2(x^2+y^2)$, and comment briefly on it physical significance.

7. **Lorentz force law**

Show that if we modify the classical Lagrangian to include the interaction of a charged particle with a magnetic field so that

$$L \rightarrow L + \frac{q}{c}\mathbf{v} \cdot \mathbf{A}(x)$$

that Lagrange's equations yield the Lorentz force law

$$m\frac{d^2\mathbf{x}}{dt^2} = \frac{q}{c}\mathbf{v} \times \mathbf{B}$$