P6572 HW #5

Due September 30, 2011

1. Show that if

$$H = \frac{\mathbf{\Pi}^2}{2m} + e\phi$$

with

$$\mathbf{\Pi} = \mathbf{p} - \frac{e}{c}\mathbf{A}$$

that if

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0$$

with probability density $\rho = |\psi|^2$ that the probability flux

$$\mathbf{j} = \frac{\hbar}{m}\mathrm{Im}(\psi^* \nabla \psi) - \frac{e}{mc}\mathbf{A}|\psi|^2$$

and that

$$\frac{d\mathbf{\Pi}}{dt} = \frac{e}{2c}\left(\frac{d\mathbf{x}}{dt} \times \mathbf{B}) - (\mathbf{B} \times \frac{d\mathbf{x}}{dt})\right) - e\nabla\phi$$

($\mathbf{\Pi}, \mathbf{p}, \mathbf{A}$, and $\mathbf{x}$ are operators.)

2. **Sakurai, p. 150, problem 37**

Consider the neutron interferometer as shown on p. 150. Prove that the difference in the magnetic fields that produce two successive maxima in the counting rates is given by

$$\Delta B = \frac{8\pi^2 \hbar c}{|e| g_n \lambda l},$$

were $g_n (= -1.91)$ is the neutron magnetic moment in units of $e - \hbar/2m_n c$.

3. **SHO Partition function**

The propagator

$$K(x', t, x, 0) = \left\langle x' \mid e^{-\frac{i}{\hbar}Ht} \mid x \right\rangle$$

If we insert the identity in the form of the complete set of energy eigenstates we get

$$\begin{aligned} K(x', t, x, 0) &= \sum_j \left\langle x' \mid e^{-\frac{i}{\hbar}Ht} \mid j \right\rangle \langle j \mid x \rangle \\ &= \sum_j \left\langle x' \mid e^{-\frac{i}{\hbar}E_j t} \mid j \right\rangle \langle j \mid x \rangle \\ &= \sum_j e^{-\frac{i}{\hbar}E_j t} \langle x' \mid j \rangle \langle j \mid x \rangle \end{aligned}$$

Set $x = x'$ and integrate over all x

$$\begin{aligned}\int_{-\infty}^{\infty} K(x,t,x,0)dx &= \sum_j \int_{-\infty}^{\infty} e^{-\frac{i}{\hbar}E_j t}\langle j \mid x\rangle\langle x \mid j\rangle dx \\ &= \sum_j e^{-\frac{i}{\hbar}E_j t}\end{aligned}$$

and then make the substitution $t \to -i\beta\hbar$

$$\int_{-\infty}^{\infty} K(x,t,x,0) = \sum_j e^{-\beta E_j}$$

We recognize the result as the partition function, where $\beta = 1/k_B T$.

The propagator for the harmonic oscillator is

$$K(x',t;x,0) = \left(\frac{m\omega}{2\pi i \sin\omega t}\right)^{\frac{1}{2}} \exp\left\{i\frac{m\omega}{2\sin\omega t}\left((x'^2 + x^2)\cos\omega t - 2xx'\right)\right\} \tag{1}$$

To determine the partition function from the propagator we follow the prescription outlined about. Put $x' = x$ and $t = -i\hbar\beta$ and integrate over all x.

$$Z = \int_{-\infty}^{\infty} K(x, -i\beta\hbar, x, 0)dx$$

(a) Make the appropriate substitutions into Equation 1 and integrate to compute the partition function for the harmonic oscillator

$$Z = \frac{e^{-\beta\hbar\omega/2}}{1 - e^{-\beta\hbar\omega}} = \sum_{j=0}^{\infty} e^{-\beta(j+\frac{1}{2})\hbar\omega}$$

(b) Show that the thermal average of the system's energy is

$$\bar{E} = \sum_i E(i)P(i) = -\frac{\partial}{\partial\beta}\ln Z$$

where $P(i) = e^{-\beta E_i}/Z$. and that

$$\bar{E} = \hbar\omega\left(\frac{1}{2} + \frac{1}{e^{\beta\hbar\omega} - 1}\right).$$

(c) Consider a crystal with N_0 atoms which for small oscillations is equivalent to $3N_0$ decoupled oscillators. Assume that all oscillators have the same frequency. The mean thermal energy of the crystal $\bar{E}_{crystal}$ is $\bar{E}$ summed over all the normal modes. Show that

$$C = \frac{1}{N_0}\frac{\partial \bar{E}_{crystal}}{\partial T} = 3k\left(\frac{\theta_E}{T}\right)^2 \frac{e^{\theta_E/T}}{(e^{\theta_E/T} - 1)^2}$$

where $\theta_E = \hbar\omega/k$ is the *Einstein temperature* and varies from crystal to crystal. Show that for $T \gg \theta_E$

$$C_{qu}(T) \to 3k$$

and for $T \ll \theta_E$

$$C_{qu} \to 3k \left(\frac{\theta_E}{T}\right)^2 e^{-\theta_E/T}$$

Although $C_{qu}(T) \to 0$ as $T \to 0$, the exponential falloff disagrees with the observed $C(T) \to T^3$ behavior. This discrepancy arises from assuming that the frequencies of all normal modes are equal, which is of course not generally true. This discrepancy was removed by Debye.

4. **Sakurai, p. 148, problem 25**

   Consider an electron confined to the *interior* of a hollow cylindrical shell whose axis coincides with the $z-$axis. The wave function is required to vanish on the inner and outer walls, $\rho = \rho_a$ and ρ_b, and also at the top and the bottom, $z = 0$ and L.

   (a) Find the energy eigenfunctions. (Do not bother with normalization.) Show that the energy eigenvalues are given by

   $$E_{lmn} = \left(\frac{\hbar^2}{2m_e}\right)\left[k_{mn}^2 + \left(\frac{l\pi}{L}\right)^2\right] \qquad (l = 1, 2, 3, ..., m = 0, 1, 2, ...),$$

   where k_{mn} is the n^{th} root of the transcendaental equation

   $$J_m(k_{mn}\rho_b)N_m(k_{mn}\rho_a) - N_m(k_{mn}\rho_b)J_m(k_{mn}\rho_a) = 0.$$

   (b) Repeat the same problem when there is a uniform magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$ for $0 < \rho < \rho_a$. Note that the energy eigenvalues are influenced by the magnetic field even though the electron never "touches" the magnetic field.

   (c) Compare, in particular, the ground state of the $B = 0$ problem with that of the $B \neq 0$ problem. Show that if we require the ground-state energy to be unchanged in the presence of B, we obtain "flux quantization"

   $$\pi\rho_a^2 B = \frac{2\pi N\hbar c}{e}, \qquad (N = 0, \pm 1, \pm 2, ...).$$

5. **Charged particle in a magnetic field**

   Consider a particle of charge e in a vector potential

   $$\mathbf{A} = \frac{B}{2}\left(-y\mathbf{i} + x\mathbf{j}\right)$$

   (a) Show that a classical particle in this potential will move in circles at an angular frequency $\omega_0 = eB/mc$.

   (b) Consider the Hamiltonian for the corresponding quantum problem

   $$H = \frac{1}{2m}\left(\Pi_x^2 + \Pi_y^2 + \Pi_z^2\right)$$

where

$$\Pi_x = p_x - e\frac{A_x}{c}, \qquad \Pi_y = p_y - e\frac{A_y}{c}, \qquad \Pi_z = p_z.$$

Show that $Q = \frac{c}{eB}\Pi_x,\ P = \Pi_y$ are canonical. Show that allowed energy levels are $E = (n+\frac{1}{2})\hbar\omega_0 + \frac{\hbar^2 k^2}{2m}$ where $\hbar k$ is the contnuous eigenvalue of the p_z operator. (Hint: Write H in terms of P and Q.)

(c) Expand H out in terms of the original variables and show as in Sakurai 35 from last week that

$$H = H(\frac{\omega_0}{2}, m) - \frac{\omega_0}{2}L_z$$

where $H(\omega_0/2, m)$ is the Hamiltonian for an isotropic two-dimensional harmonic oscillator. Argue that the same basis that diagonalized $H(\omega_0/2, m)$ will diagonalize H. By thinking in terms of this basis, show that the allowed levels for H are $E = (k + \frac{1}{2}|m| - \frac{1}{2}m + \frac{1}{2})\hbar\omega_0$, where k is any integer and m is the angular momentum, where we are neglecting the motion in the z-direction. Convince yourself that you get the same levels from this formula as from the earlier one $[E = (n + 1/2)\hbar\omega_0]$.