Problem 1

We wish to determine the properties of the two-by-two matrix:

$$U = \frac{a_0 + i\boldsymbol{\sigma}\cdot\boldsymbol{a}}{a_0 - i\boldsymbol{\sigma}\cdot\boldsymbol{a}}$$

where a_0 is a real number and $\boldsymbol{a}$ is a real three-vector. We define $A \equiv a_0 + i\boldsymbol{\sigma}\cdot\boldsymbol{a}$. Thus,

$$A = \begin{pmatrix} a_0 + i a_3 & a_2 + i a_1 \\ -a_2 + i a_1 & a_0 - i a_3 \end{pmatrix}$$

so $\det A = |a_0 + i a_3|^2 + |a_2 + i a_1|^2 = a_0^2 + |\boldsymbol{a}|^2 = \det A^\dagger$. Consider the product:

$$A A^\dagger = (a_0 + i\boldsymbol{\sigma}\cdot\boldsymbol{a})(a_0 - i\boldsymbol{\sigma}\cdot\boldsymbol{a}) = a_0^2 + (\boldsymbol{\sigma}\cdot\boldsymbol{a})^2 = a_0^2 + |\boldsymbol{a}|^2 = \det A$$

where we use the well known properties $\boldsymbol{\sigma}^\dagger = \boldsymbol{\sigma}$ and $(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}})^2 = 1$ for any unit vector $\hat{\boldsymbol{n}}$. Thus, $A^{-1} = \frac{1}{\det A} A^\dagger$. We find:

$$U = A\,(A^\dagger)^{-1} = A\,(A^{-1})^\dagger = \frac{1}{\det A} A^2$$

Thus, $U U^\dagger = \frac{1}{(\det A)^2} A^2 (A^\dagger)^2 = \frac{1}{\det A} A A^\dagger = 1 = U^\dagger U$, so U is unitary. Furthermore, $\det U = \frac{1}{(\det A)^2} \det A^2 = 1$, so U is unimodular. Stated differently, $U \in \mathrm{SU}(2)$.

We now write out explicitly:

$$U = \frac{1}{\det A} A^2 = \frac{1}{\det A}(a_0 + i\boldsymbol{\sigma}\cdot\boldsymbol{a})^2 = \frac{1}{\det A}\left(a_0^2 + 2 i a_0 \boldsymbol{\sigma}\cdot\boldsymbol{a} - |\boldsymbol{a}|^2\right) = \frac{a_0^2 - |\boldsymbol{a}|^2}{a_0^2 + |\boldsymbol{a}|^2} + \frac{2 a_0 |\boldsymbol{a}|}{a_0^2 + |\boldsymbol{a}|^2}(i\boldsymbol{\sigma}\cdot\hat{\boldsymbol{a}})$$

where $\hat{\boldsymbol{a}} \equiv \boldsymbol{a}/|\boldsymbol{a}|$. We recognize the form $\exp\left(\frac{-i\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}\,\phi}{2}\right) = \cos\frac{\phi}{2} - i\sin\frac{\phi}{2}\,\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}$ (Sakurai eqn. 3.2.44), where

$$\begin{aligned} \cos\frac{\phi}{2} &= \frac{a_0^2 - |\boldsymbol{a}|^2}{a_0^2 + |\boldsymbol{a}|^2} \\ \sin\frac{\phi}{2} &= -\frac{2 a_0 |\boldsymbol{a}|}{a_0^2 + |\boldsymbol{a}|^2} \\ \hat{\boldsymbol{n}} &= \hat{\boldsymbol{a}} \end{aligned}$$

Thus, U represents a rotation by an angle $\phi = -2\sin^{-1}\left(\frac{2 a_0 |\boldsymbol{a}|}{a_0^2 + |\boldsymbol{a}|^2}\right)$ about the axis $\hat{\boldsymbol{a}}$, where the ambiguity in the value of $\sin^{-1}$ can be resolved by looking at the cos term.

Problem 2

a)

Recall the angular momentum algebra:

$$[J_x, J_y] = i\hbar J_z\ ,\ [J_y, J_z] = i\hbar J_x\ ,\ [J_z, J_x] = i\hbar J_y$$

Given a basis $|i\rangle$, the matrix representations of the J_x is $[J_x]_{ij} \equiv \langle i|J_x|j\rangle$, and likewise for J_y and J_z. Thus, since J_x and J_z have real matrix elements, we find:

$$\langle i|J_x|j\rangle = \langle j|J_x|i\rangle \quad , \quad \langle i|J_z|j\rangle = \langle j|J_z|i\rangle$$

where we use the Hermicity of J_x and J_z. Taking the matrix elements of the third commutator given above, we find

$$\langle i|[J_x, J_z]|j\rangle = i\hbar\,\langle i|J_y|j\rangle$$

where

$$\begin{aligned}\langle i|[J_x, J_z]|j\rangle &= \sum_k \big(\langle i|J_x|k\rangle\langle k|J_z|j\rangle - \langle i|J_z|k\rangle\langle k|J_x|j\rangle\big) \\ &= \sum_k \big(\langle j|J_z|k\rangle\langle k|J_x|i\rangle - \langle j|J_x|k\rangle\langle k|J_z|i\rangle\big) \\ &= -\langle j|[J_x, J_z]|i\rangle\end{aligned}$$

Therefore,

$$\langle i|J_y|j\rangle = -\langle j|J_y|i\rangle$$

so, since J_y is Hermitean, J_y has imaginary matrix elements.

b)

Suppose that $[\mathcal{O}, J_x] = [\mathcal{O}, J_y] = 0$ for some operator $\mathcal{O}$. Therefore, we find:

$$[\mathcal{O}, [J_x, J_y]] = -[J_y, [\mathcal{O}, J_x]] - [J_x, [J_y, \mathcal{O}]] = 0$$

where we apply the Jacobi identity, $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$. Applying the above result to the angular momentum algebra $[J_x, J_y] = i\hbar J_z$, we conclude that

$$[\mathcal{O}, J_z] = 0$$

so the result is proven.

c)

We have

$$R_{\boldsymbol{n}}(\theta) \equiv e^{-i\theta\boldsymbol{n}\cdot\boldsymbol{J}/\hbar}$$

where $\boldsymbol{n}$ is a unit vector and $R_{\boldsymbol{n}}(\theta)$ represents a (right-handed) rotation by an angle θ about the axis defined by $\boldsymbol{n}$. We wish to compute

$$\hat{R} = R_{\boldsymbol{v}}^{-1}(\varepsilon)\,R_{\boldsymbol{u}}^{-1}(\varepsilon)\,R_{\boldsymbol{v}}(\varepsilon)\,R_{\boldsymbol{u}}(\varepsilon) = e^{i\varepsilon\boldsymbol{v}\cdot\boldsymbol{J}/\hbar}\,e^{i\varepsilon\boldsymbol{u}\cdot\boldsymbol{J}/\hbar}\,e^{-i\varepsilon\boldsymbol{v}\cdot\boldsymbol{J}/\hbar}\,e^{-i\varepsilon\boldsymbol{u}\cdot\boldsymbol{J}.\hbar}$$

where $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ form a right-handed coordinate system ($\boldsymbol{u} \times \boldsymbol{v} = \boldsymbol{w}$, etc), and ε is an infinitesimal angle.

To solve this problem, we will use the identity

$$e^{\varepsilon A} e^{\varepsilon B} = e^{\varepsilon A + \varepsilon B + \frac{1}{2}\varepsilon^2 [A,B] + \mathcal{O}(\varepsilon^3)}$$

Multiplying on the left by $e^{-\varepsilon B}$ and applying the original identity to the RHS, we find:

$$e^{-\varepsilon B} e^{\varepsilon A} e^{\varepsilon B} = e^{-\varepsilon B} e^{\varepsilon A + \varepsilon B + \frac{1}{2}\varepsilon^2 [A,B] + \mathcal{O}(\varepsilon^3)} = e^{\varepsilon A + \varepsilon^2 [A,B] + \mathcal{O}(\varepsilon^3)}$$

Multiplying on the left by $e^{-\varepsilon A}$ and applying the original identity once more, we find:

$$\varepsilon^{-\varepsilon A} e^{-\varepsilon B} e^{\varepsilon A} e^{\varepsilon B} = e^{\varepsilon^2 [A,B] + \mathcal{O}(\varepsilon^3)}$$

This can also be written as:

$$e^{\varepsilon A} e^{\varepsilon B} = e^{\varepsilon B} e^{\varepsilon A} e^{\varepsilon^2 [A,B] + \mathcal{O}(\varepsilon^3)}$$

where the higher-order terms are not present if $[A, B]$ commutes with A and B. For the problem at hand, we put in $A = -i\boldsymbol{v} \cdot \boldsymbol{J}/\hbar$ and $B = -i\boldsymbol{u} \cdot \boldsymbol{J}/\hbar$, so that

$$[A, B] = -\frac{1}{\hbar^2} v_i u_j [J_i, J_j] = \frac{i}{\hbar} \varepsilon_{ijk} u_i v_j J_k = \frac{i}{\hbar} \boldsymbol{w} \cdot \boldsymbol{J}$$

where the last step follows from $\boldsymbol{u} \times \boldsymbol{v} = \boldsymbol{w}$, or $\varepsilon_{ijk} u_i v_j = w_k$, and we use the commutator

$$[J_i, J_j] = i\hbar \varepsilon_{ijk} J_k$$

Therefore, applying the identity we derived above, we find

$$\hat{R} = \exp\left[\frac{i}{\hbar} \varepsilon^2 \boldsymbol{w} \cdot \boldsymbol{J} + \mathcal{O}\big(\varepsilon^3\big)\right] = R_{\boldsymbol{w}}\big(-\varepsilon^2\big) + \mathcal{O}\big(\varepsilon^3\big)$$

which is the desired result.

Problem 3

In this problem, we study some of the properties of positronium, a bound state of an electron and a positron. We focus on the spin Hamiltonian, ignoring the spatial wavefunction.

Written out explicitly, we have:

$$H = A\left[S_x^{(1)} S_x^{(2)} + S_y^{(1)} S_y^{(2)} + S_z^{(1)} S_z^{(2)}\right] + \left(\frac{eB}{mc}\right)\left(S_z^{(1)} - S_z^{(2)}\right)$$

where $S^{(1)}$ is the spin operator for the electron and $S^{(2)}$ is the spin operator for the positron. We consider the state $|+^{(1)}, -^{(2)}\rangle$, where $\pm^{(1)}$ refer to the $\pm\hbar/2$ $S_z^{(1)}$ eigenstates, and likewise for $\pm^{(2)}$.

a)

We set $A=0$. Apply H to the state in question, we find:

$$\begin{aligned} H\,|+^{(1)},-^{(2)}\rangle &= \left(\frac{eB}{mc}\right)\left(\frac{\hbar}{2}-\left(-\frac{\hbar}{2}\right)\right)|+^{(1)},-^{(2)}\rangle \\ &= \frac{eB\hbar}{mc}\,|+^{(1)},-^{(2)}\rangle \end{aligned}$$

Thus, $|+^{(1)},-^{(2)}\rangle$ is an energy eigenstate with energy $\frac{eB\hbar}{mc}$.

b)

Now we set $B=0$. We have:

$$\begin{aligned} S_x\,|\pm\rangle &= \frac{\hbar}{2}\,|\mp\rangle \\ S_y\,|\pm\rangle &= \pm i\frac{\hbar}{2}\,|\mp\rangle \end{aligned}$$

Thus,

$$\begin{aligned} H\,|+^{(1)},-^{(2)}\rangle &= \frac{A\hbar^2}{4}\left[|-^{(1)},+^{(2)}\rangle + i(-i)\,|-^{(1)},+^{(2)}\rangle - |+^{(1)},-^{(2)}\rangle\right] \\ &= \frac{A\hbar^2}{4}\left[2\,|-^{(1)},+^{(2)}\rangle - |+^{(1)},-^{(2)}\rangle\right] \end{aligned}$$

Clearly, $|+^{(1)},-^{(2)}\rangle$ is not an eigenstate. We compute:

$$\begin{aligned} \langle H\rangle &= \langle +^{(1)},-^{(2)}|H\,|+^{(1)},-^{(2)}\rangle \\ &= -\frac{A\hbar^2}{4} \end{aligned}$$

Thus, the energy expectation value for this state is $-A\hbar^2/4$.

In fact, it is straightforward to check that the eigenstates are

$$|+^{(1)},+^{(2)}\rangle, |-^{(1)},-^{(2)}\rangle, \frac{1}{\sqrt{2}}(|+^{(1)},-^{(2)}\rangle + |-^{(1)},+^{(2)}\rangle) \text{ and } \frac{1}{\sqrt{2}}(|+^{(1)},-^{(2)}\rangle - |-^{(1)},+^{(2)}\rangle).$$

where the first three are degenerate with eigenenergy $A\hbar^2/4$, and the last is the ground state, with eigenenergy $-3A\hbar^2/4$. This three-one splitting should be familiar. The first three states form an $l=1$ angular momentum triplet, and the last is the $l=0$ singlet. The energy splitting between them is the positronium analog of the hyperfine splitting of the hydrogen ground state.

Problem 4

Consider the Hamiltonian:

$$H = \frac{1}{2}\sum_i I_i^{-1} K_i^2$$

where K_i are the angular momentum operators, satisfying the usual commutation relation:

$$[K_i, K_j] = i\hbar\,\varepsilon_{ijk}\,K_k$$

The Heisenberg equation of motion is:

$$\begin{aligned}
\dot{K}_i &= \frac{1}{i\hbar}[K_i, H] \\
&= \frac{1}{2i\hbar}\left[K_i, \sum_j I_j^{-1} K_j^2\right] \\
&= \frac{1}{2i\hbar}\sum_j I_j^{-1}\left([K_i, K_j]\,K_j + K_j\,[K_i, K_j]\right) \\
&= \frac{1}{2}\sum_{j,k} \varepsilon_{ijk}\,I_j^{-1}\,(K_k\,K_j + K_j\,K_k) \\
&= \frac{1}{2}\sum_{j,k} \varepsilon_{ijk}\left(I_j^{-1} - I_k^{-1}\right) K_j\,K_k
\end{aligned}$$

We now take the classical limit, $\hbar \to 0$. Thus, $[K_i, K_j] \to 0$. Writing out the sum, we find:

$$\dot{K}_1 = \left(I_2^{-1} - I_3^{-1}\right) K_2\,K_3$$

and cyclic permutations. We have $K_i = I_i\,\omega_i$. Thus,

$$\begin{aligned}
I_1\,\dot{\omega}_1 &= -(I_2 - I_3)\,\omega_2\,\omega_3 \\
I_2\,\dot{\omega}_2 &= -(I_3 - I_1)\,\omega_3\,\omega_1 \\
I_3\,\dot{\omega}_3 &= -(I_1 - I_2)\,\omega_1\,\omega_2
\end{aligned}$$

and cyclic permutations. These are Euler's equations (modulo a sign).

Problem 5

We have

$$\mathcal{D}^{(1/2)}(\alpha, \beta, \gamma) = \begin{pmatrix} e^{-i(\alpha+\gamma)/2}\cos\beta/2 & -e^{-i(\alpha-\gamma)/2}\sin\beta/2 \\ e^{i(\alpha-\gamma)/2}\sin\beta/2 & e^{i(\alpha+\gamma)/2}\cos\beta/2 \end{pmatrix}$$

As in problem 1, we want to rewrite this in the form

$$\exp\left(\frac{-i\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}\,\phi}{2}\right) = \cos\left(\frac{\phi}{2}\right) - i\sin\left(\frac{\phi}{2}\right)\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}$$

In particular, we wish to find ϕ, the angle of rotation. As a shortcut, we take the trace of both expressions. Since $\mathrm{Tr}\,\boldsymbol{\sigma} = 0$, we find:

$$\begin{aligned}
2\cos\left(\frac{\phi}{2}\right) &= \left[e^{i(\alpha+\gamma)/2} + e^{-i(\alpha+\gamma)/2}\right]\cos\beta/2 \\
&= 2\cos\left(\frac{\alpha+\gamma}{2}\right)\cos\left(\frac{\beta}{2}\right)
\end{aligned}$$

Thus,

$$\phi \;=\; 2\cos^{-1}\left[\cos\left(\frac{\alpha+\gamma}{2}\right)\cos\left(\frac{\beta}{2}\right)\right]$$

This is sufficient to determine ϕ up to its sign, which is meaningless in any case unless we specify the sign of the axis of rotation.

To find the axis of rotation, we use

$$a_0 - i\boldsymbol{\sigma}\cdot\boldsymbol{a} \;=\; \begin{pmatrix} a_0 - i a_3 & -a_2 - i a_1 \\ a_2 - i a_1 & a_0 + i a_3 \end{pmatrix}$$

from problem 1. Thus,

$$\sin\left(\frac{\phi}{2}\right)\hat{\boldsymbol{n}} \;=\; \left(-\sin\left(\frac{\alpha-\gamma}{2}\right)\sin\left(\frac{\beta}{2}\right),\; \cos\left(\frac{\alpha-\gamma}{2}\right)\sin\left(\frac{\beta}{2}\right),\; \sin\left(\frac{\alpha+\gamma}{2}\right)\cos\left(\frac{\beta}{2}\right)\right)$$

Problem 6

We start with the Hamiltonian

$$H_0 \;=\; \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix} \;=\; \begin{pmatrix} E_1 - \frac{i\hbar\gamma_1}{2} & 0 \\ 0 & E_2 \end{pmatrix}$$

where E_1, E_2 and γ_1 are real and $\gamma_1 > 0$. The state $|1\rangle$ is unstable with lifetime $1/\gamma_1$, and the state $|2\rangle$ is stable. We now perturb the Hamiltonian by coupling $|1\rangle$ to $|2\rangle$:

$$H \;=\; H_0 + W \;=\; \begin{pmatrix} \epsilon_1 & V \\ V^\star & \epsilon_2 \end{pmatrix}$$

where $W_{11} = W_{22} = 0$ and $W_{12} = W_{21}^\star = V$. The coupling between $|1\rangle$ and $|2\rangle$ will render $|2\rangle$ unstable, as we now show.

a)

We find the eigenvalues of H. The eigenvalue equation gives:

$$(\epsilon_1 - \lambda)(\epsilon_2 - \lambda) \;=\; |V|^2$$

Thus,

$$\lambda^2 - (\epsilon_1 + \epsilon_2)\,\lambda + \left(\epsilon_1\,\epsilon_2 - |V|^2\right) \;=\; 0$$

so

$$\begin{aligned}\lambda \;&=\; \frac{\epsilon_1 + \epsilon_2 \pm \sqrt{(\epsilon_1+\epsilon_2)^2 - 4\,(\epsilon_1\,\epsilon_2 - |V|^2)}}{2} \\ &=\; \frac{\epsilon_1+\epsilon_2}{2} \pm \sqrt{\left(\frac{\epsilon_1 - \epsilon_2}{2}\right)^2 + |V|^2}\end{aligned}$$

Suppose that $2\,|V| \ll |\epsilon_1 - \epsilon_2| = \sqrt{(E_1 - E_2)^2 + \frac{\hbar^2\,\gamma_1^2}{4}}$. In this case, we can approximate:

$$\pm\sqrt{\left(\frac{\epsilon_1-\epsilon_2}{2}\right)^2 + |V|^2} \;\simeq\; \pm\frac{\epsilon_1-\epsilon_2}{2}\left(1 + \frac{1}{2}\,4\,|V|^2/(\epsilon_1-\epsilon_2)^2\right)$$

Therefore,

$$\begin{aligned}\lambda_1 &\simeq \epsilon_1 + \frac{|V|^2}{\epsilon_1-\epsilon_2} = E_1 - \frac{i\hbar\gamma_1}{2} + \frac{|V|^2}{E_1 - i\hbar\gamma_1/2 - E_2}\\ \lambda_2 &\simeq \epsilon_2 - \frac{|V|^2}{\epsilon_1-\epsilon_2} = E_2 - \frac{|V|^2}{E_1 - i\hbar\gamma_1/2 - E_2}\end{aligned}$$

For $|V| \neq 0$, both λ_1 and λ_2 will be complex, so both states become unstable.

b)

We write:

$$\lambda_1 = \Delta_1 - \frac{i\hbar\Gamma_1}{2}$$

and similarly for λ_2. In the weak coupling limit, $2\,|V| \ll \sqrt{(E_1 - E_2)^2 + \frac{\hbar^2\,\gamma_1^2}{4}}$, we find:

$$\begin{aligned}\lambda_1 &\simeq E_1 - \frac{i\hbar\gamma_1}{2} + \frac{|V|^2\,(E_1 - E_2 + i\hbar\gamma_1/2)}{(E_1-E_2)^2 + \hbar^2\,\gamma_1^2/4}\\ \lambda_2 &\simeq E_2 - \frac{|V|^2\,(E_1 - E_2 + i\hbar\gamma_1/2)}{(E_1-E_2)^2 + \hbar^2\,\gamma_1^2/4}\end{aligned}$$

Collecting terms, we read off:

$$\begin{aligned}\Delta_1 &\simeq E_1 + \frac{|V|^2\,(E_1-E_2)}{(E_1-E_2)^2 + \hbar^2\,\gamma_1^2/4} \qquad \Gamma_1 \simeq \gamma_1 - \frac{|V|^2\,\gamma_1}{(E_1-E_2)^2 + \hbar^2\,\gamma_1^2/4}\\ \Delta_2 &\simeq E_2 - \frac{|V|^2\,(E_1-E_2)}{(E_1-E_2)^2 + \hbar^2\,\gamma_1^2/4} \qquad \Gamma_2 \simeq \frac{|V|^2\,\gamma_1}{(E_1-E_2)^2 + \hbar^2\,\gamma_1^2/4}\end{aligned}$$

c)

Now we want to move beyond the weak coupling approximation. We set $E_1 = E_2$. The eigenvalues become:

$$\begin{aligned}\lambda &= E_1 + \frac{1}{2}(-i\hbar\gamma_1/2) \pm \sqrt{\left(\frac{-i\hbar\gamma_1/2}{2}\right)^2 + |V|^2}\\ &= E_1 - i\hbar\gamma_1/4 \pm \sqrt{|V|^2 - (\hbar\gamma_1/4)^2}\end{aligned}$$

We assume that $|V| > \hbar\gamma_1/4$. Thus, the square root is real, and we find:

$$\begin{aligned}\Delta_+ &= E_1 + \sqrt{|V|^2 - (\hbar\gamma_1/4)^2}\\ \Delta_+ &= E_1 - \sqrt{|V|^2 - (\hbar\gamma_1/4)^2}\\ \Gamma_+ &= \gamma_1/2 = \Gamma_-\end{aligned}$$

where $\lambda_\pm = E_1 \pm \sqrt{|V|^2 - (\hbar\gamma_1/4)^2} - i\hbar\gamma_1/4$. We find the corresponding eigenvectors:

$$\begin{pmatrix} E_1 - i\hbar\gamma_1/2 - \lambda_\pm & V \\ V^\star & E_1 - \lambda_\pm \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0$$

Thus,

$$V^\star a + (E_1 - \lambda_\pm)\, b = 0$$

so $b = \frac{V^\star a}{\lambda_\pm - E_1} = \frac{V^\star a}{-i\hbar\gamma_1/4 \pm \sqrt{|V|^2 - (\hbar\gamma_1/4)^2}}$. Therefore, the eigenvectors are:

$$|\pm\rangle = \begin{pmatrix} 1 \\ \frac{V^\star}{-i\hbar\gamma_1/4 \pm \sqrt{|V|^2 - (\hbar\gamma_1/4)^2}} \end{pmatrix}$$

(it is not necessary to normalize them; moreover, since H is not Hermitean $\langle + | - \rangle \neq 0$ in general). We rewrite $|2\rangle$ in this basis:

$$|2\rangle = \frac{V}{2\sqrt{|V|^2 - (\hbar\gamma_1/4)^2}} \left(|+\rangle - |-\rangle\right)$$

Thus, the transition amplitude is given by:

$$\begin{aligned} \langle 1|\, e^{-iHt/\hbar} |2\rangle &= \langle 1| \frac{V}{2\sqrt{|V|^2 - (\hbar\gamma_1/4)^2}} \left(e^{-i\Delta_+ t/\hbar - \Gamma t/2} \Big| + \rangle - e^{-i\Delta_- t/\hbar - \Gamma t/2} | - \rangle\right) \\ &= \frac{V}{\sqrt{|V|^2 - (\hbar\gamma_1/4)^2}}\, e^{-\Gamma t/2} \frac{1}{2}\left(e^{-i\Delta_+ t/\hbar} - e^{-i\Delta_- t/\hbar}\right) \\ &= \frac{-iV}{\sqrt{|V|^2 - (\hbar\gamma_1/4)^2}}\, e^{-\Gamma t/2}\, e^{-iE_1 t/\hbar} \sin\left(\frac{\sqrt{|V|^2 - (\hbar\gamma_1/4)^2}\, t}{\hbar}\right) \end{aligned}$$

Therefore, the probability to find the system in state $|1\rangle$ as a function of time is:

$$\begin{aligned} P_1(t) &= |\langle 1|\, e^{-iHt/\hbar} |2\rangle|^2 \\ &= \frac{|V|^2}{|V|^2 - (\hbar\Gamma/2)^2}\, e^{-\Gamma t} \sin^2\left(\frac{\sqrt{|V|^2 - (\hbar\Gamma/2)^2}\, t}{\hbar}\right) \end{aligned}$$

where $\Gamma = \gamma_1/2$ is the inverse lifetime of both eigenstates.

d)

As in part (c), we set $E_1 = E_2$, so that

$$\lambda = E_1 - i\hbar\gamma_1/4 \pm \sqrt{|V|^2 - (\hbar\gamma_1/4)^2}$$

Now we assume that $|V| < (\hbar\gamma_1/4)$. The above expression can be rewritten:

$$\begin{aligned} \lambda_\pm &= E_1 - i\hbar\gamma_1/4 \mp i\sqrt{(\hbar\gamma_1/4)^2 - |V|^2} \\ &= E_1 - i\hbar\gamma_1/4\left(1 \pm \sqrt{1 - (4|V|/\hbar\gamma_1)^2}\right) \end{aligned}$$

Thus, $\Delta_+ = \Delta_- = E_1$, and

$$\Gamma_+ \;=\; \frac{\gamma_1}{2}\left(1+\sqrt{1-(4\,|V|/\hbar\gamma_1)^2}\right) \quad \Gamma_- \;=\; \frac{\gamma_1}{2}\left(1-\sqrt{1-(4\,|V|/\hbar\gamma_1)^2}\right)$$

Now the eigenstate $|+\rangle$ is shorter lived than the eigenstate $|-\rangle$. We find the eigenvectors as above:

$$|\pm\rangle \;=\; \left(\begin{array}{c} 1 \\ \frac{V^\star}{-\,i\hbar\gamma_1/4\left(1\pm\sqrt{1-(4\,|V|/\hbar\gamma_1)^2}\right)} \end{array}\right)$$

Thus,

$$|2\rangle \;=\; \frac{2iV}{\hbar\gamma_1\sqrt{1-(4\,|V|/\hbar\gamma_1)^2}}\,(|+\rangle - |-\rangle)$$

The transition amplitude is therefore:

$$\begin{aligned}\langle 1|e^{-iHt/\hbar}|2\rangle \;&=\; \langle 1|\frac{2iV}{\hbar\gamma_1\sqrt{1-(4\,|V|/\hbar\gamma_1)^2}}\,e^{-iE_1t/\hbar}\left(e^{-\Gamma_+t/2}\Big|+\rangle - e^{-\Gamma_-t/2}|-\rangle\right) \\ &=\; \frac{2iV}{\hbar\gamma_1\sqrt{1-(4\,|V|/\hbar\gamma_1)^2}}\,e^{-iE_1t/\hbar}\left(e^{-\Gamma_+t/2} - e^{-\Gamma_-t/2}\right) \\ &=\; \frac{-\,4iV}{\hbar\gamma_1\sqrt{1-(4\,|V|/\hbar\gamma_1)^2}}\,e^{-iE_1t/\hbar}\,e^{-\gamma_1t/4}\sinh\left(\frac{(\Gamma_+-\Gamma_-)\,t}{4}\right)\end{aligned}$$

Thus,

$$\begin{aligned}P_1(t) \;&=\; |\langle 1|\,e^{-iHt/\hbar}|2\rangle|^2 \\ &=\; \frac{|V|^2}{(\hbar\gamma_1/4)^2-|V|^2}\,e^{-\gamma_1t/2}\sinh^2\left(\gamma_1\sqrt{1-(4\,|V|/\hbar\gamma_1)^2}\,t\Big/4\right)\end{aligned}$$

e)

For $|W| > \hbar\gamma_1/4$, the mixing is faster than the decay, and the system oscillates back and forth between the states $|1\rangle$ and $|2\rangle$. This is analogous to an underdamped mechanical system.

For $|W| < \hbar\gamma_1/4$, the decay rate dominates, and no oscillations occur. Instead, the system evolves towards the longer lived state $|-\rangle$, with some mixture of $|1\rangle$ and $|2\rangle$, at the same time as the overall amplitude decreases. This is analogous to an overdamped mechanical system.

The case $|W| = \hbar\gamma_1/4$ is more difficult to solve, since H is no longer diagonalizable (this can occur, since H is not Hermitean). One might guess that this case is analogous to a critically damped mechanical system.

Problem 7

Before solving this problem, let me review some background on the neutral Kaon system. The states $|K_0\rangle$ and $|\bar{K}_0\rangle$ are psuedoscalar[1] mesons with quark constituents $d\,\bar{s}$ and $\bar{d}\,s$. These mesons can be produced by interactions due to the strong force in particle colliders or by cosmic rays hitting the upper atmosphere. Their rest mass is 497.614(24) MeV (PDG). They are unstable due to the weak force, and decay predominantly to either two or three pions (either charged or neutral), or semileptonically (which we will ignore).

1. That is, they have zero spin and odd intrinsic parity.

We choose their relative phases so that $C|K_0\rangle = -|\bar{K}_0\rangle$ and $C|\bar{K}_0\rangle = -|K_0\rangle$ where C is the charge conjugation operator. Thus, $CP|K_0\rangle = |\bar{K}_0\rangle$ and vice versa, where P is the parity operator. Unlike C and P, the combination CP is conserved by the weak interaction to a good approximation. Thus, it is convenient to work with the basis:

$$|K_1\rangle = \frac{1}{\sqrt{2}}\left(|K_0\rangle + |\bar{K}_0\rangle\right)$$

$$|K_2\rangle = \frac{1}{\sqrt{2}}\left(|K_0\rangle - |\bar{K}_0\rangle\right)$$

These are, respectively, even and odd eigenstates of CP. CP invariance dictates that $|K_1\rangle$ can only decay to an even eigenstate of CP, whereas $|K_2\rangle$ can only decay to an odd eigenstate. As it turns out, this means that the leading decay mode for $|K_1\rangle$ is two pions, whereas for $|K_2\rangle$ it is either three pions or a three-body semileptonic decay. Three-body decays are kinematically suppressed, so the state $|K_2\rangle$ is much longer lived that $|K_1\rangle$. Particle physicists refer to these two states as K_L and K_S, with lifetimes $5.116(20) \times 10^{-8}$ s and $0.8953(5) \times 10^{-10}$ s respectively.

Consider the Hamiltonian for the neutral kaon system in the $|K_1\rangle$, $|K_2\rangle$ basis:

$$H_0 = \begin{pmatrix} E_1 - i\hbar\gamma_1/2 & 0 \\ 0 & E_2 - i\hbar\gamma_2/2 \end{pmatrix}$$

The approximate CP symmetry of the weak interaction forbids off-diagonal couplings. To a good approximation, $E_1 = E_2$,[2] and $\gamma_1 \gg \gamma_2$. Thus, we approximate the Hamiltonian as:

$$H_0 \simeq \begin{pmatrix} E - i\hbar\gamma_1/2 & 0 \\ 0 & E \end{pmatrix}$$

In this picture, the K_L is approximately stable, whereas the K_S is unstable, and rapidly decays.

If the gravitational mass of $|K_0\rangle$ and $|\bar{K}_0\rangle$ is different, this will introduce off-diagonal couplings into the Hamiltonian:

$$\begin{aligned} H_{\text{int}} &= -\frac{GM}{R}\left(m_{K_0}|K_0\rangle\langle K_0| + m_{\bar{K}_0}|\bar{K}_0\rangle\langle\bar{K}_0|\right) \\ &= -\frac{GM\left(m_{K_0} + m_{\bar{K}_0}\right)}{2R}\left(|K_0\rangle\langle K_0| + |\bar{K}_0\rangle\langle\bar{K}_0|\right) - \frac{GM\left(m_{K_0} - m_{\bar{K}_0}\right)}{2R}\left(|K_0\rangle\langle K_0| - |\bar{K}_0\rangle\langle\bar{K}_0|\right) \end{aligned}$$

The first term just shifts the overall energy, and can be dropped. The second term can be reexpressed in the $|K_1\rangle$, $|K_2\rangle$ basis as:

$$H_{\text{int}} = \begin{pmatrix} 0 & -\frac{GM\Delta m}{2R} \\ -\frac{GM\Delta m}{2R} & 0 \end{pmatrix}$$

where $\Delta m \equiv m_{K_0} - m_{\bar{K}_0}$. Let's estimate the size of the coupling term under the maximal assumption $\Delta m = 2m_{K_0}$. We have $GM_E/R_E = (GM_E/R_E^2)\,R_E = g_E\,R_E = (9.8\ \text{m/s}^2) \times (6.4 \times 10^6\ \text{m}) = 6.3 \times 10^7\ \text{m}^2/\text{s}^2 = 7.0 \times 10^{-10}\,c^2$. Thus, $V = \frac{GM\Delta m}{2R} \simeq \left(\frac{7}{2} \times 10^{-10}\right)\Delta m c^2$. Putting in $\Delta m \simeq 975\ \text{MeV}/c^2$, we find $V \simeq 0.34$ eV. By comparison, $\hbar\gamma_1/4 \simeq (6.582 \times 10^{-16}\ \text{eV}\cdot\text{s})/(4 \times 0.8953 \times 10^{-10}\,\text{s}) \simeq 1.84 \times 10^{-6}\,\text{eV}$. Thus, $V \gg \hbar\gamma_1/4$, and we are in the "overdamped" regime, as described in problem 5. In this case, K_S and K_L would have equal lifetimes and large intermixing, with $\Gamma^{-1} = (\gamma_1/2)^{-1} \simeq 1.8 \times 10^{-10}$ s.

2. In fact, $E_2 - E_1 = 3.483(6) \times 10^{-12}$ MeV, corresponding to a small off diagonal term in the K_0, $\bar{K}_0$ basis. This very small value is due to the fact that strong and electromagnetic interactions conserve quark flavor. The weak interaction does not conserve flavor, and gives rise to this mixing. An observable consequence is that K_0's oscillate in $\bar{K}_0$'s and vice versa.

Obviously, to be consistent with experiment, Δm must be much smaller than this. At a minimum, K_L and K_S have different lifetimes, so we must have $V < \hbar\gamma_1/4 = 1.84 \times 10^{-6}$ eV. This translates into the limit

$$\Delta m \ < \ \frac{1.84 \times 10^{-6}\,\mathrm{eV}}{3.5 \times 10^{-10}\, c^2} \ = \ 5.3 \text{ KeV}$$

Thus, the gravitational mass of the K_0 and $\bar{K}_0$ must match to one part in 10^5!

At this point, several corrections that we have so far negelected will become important. The energy splitting $E_1 - E_2 = 3.483(6) \times 10^{-6}$ eV is now comparable to the off-diagonal term, so we cannot set $E_1 = E_2$ any longer. However, we haven't yet harnessed the full set of experimental data on this system either. We know that K_L is much longer lived than K_S, so (in the language of problem 5) the system must be far into the overdamped (weakly coupled) regime. In fact, experiments show that the K_L decays to two pions with a branching ratio of about 2×10^{-3}. This is understood to occur due to CP violaton in the weak interaction. Δm must be sufficiently small that the additional mixing between K_1 and K_2 does not lead to two pion decays in excess of the measured value. Accounting for all of these effects is beyond the scope of this problem.