P6572 HW #7
Due October 14, 2011

1. A simultaneous eigenstate of J^2 and J_z is denoted $| \, j, m\rangle$.

   (a) Show that the expectation values of the operators J_x and J_y for this state are zero.

   (b) Show that if any operator commutes with 2 components of an angular momentum operator, it must commute with the third component.

2. We would like to study the evolution of a state that starts out as $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and is subject to the $\mathbf{B}$ field

$$\mathbf{B} = B\cos\omega t\hat{\mathbf{i}} - B\sin\omega t\hat{\mathbf{j}} + B_0\hat{\mathbf{k}} \qquad (B \ll B_0)$$

This state obeys

$$i\hbar\frac{d}{dt}| \, \psi(t)\rangle = H| \, \psi\rangle \tag{1}$$

where $H = -\gamma\mathbf{S}\cdot\mathbf{B}$ and $\mathbf{B}$ is time dependent. Since classical reasoning suggests that in a frame rotating at frequency $(-\omega\hat{\mathbf{k}})$ the Hamiltonian should be time independent and governed by

$$\mathbf{B}_r = B\hat{\mathbf{i}}_{\mathbf{r}} + (B_0 - \omega/\gamma)\hat{\mathbf{k}}$$

consider the ket in the rotating frame, $| \, \psi_r(t)\rangle$, related to $| \, \psi(t)\rangle$ by a rotation angle ωt:

$$| \, \psi_r(t)\rangle = e^{-i\omega t S_z/\hbar}| \, \psi(t)\rangle. \tag{2}$$

($\mathbf{B}_r$ is the effective field in the rotating frame. $\hat{\mathbf{i}}_{\mathbf{r}}$ is the unit vector in the x-direction in the rotating frame.)

Combine Equations 1 and 2 to derive Schrodinger's equation for $| \, \psi_r(t)\rangle$ in the S_z basis and verify that the classical expectation is borne out. Solve for $| \, \psi_r(t)\rangle = U_r(t)| \, \psi_r(0)\rangle$ by computing $U_r(t)$, the propagator in the rotating frame. Rotate back to the lab and show that in the S_z basis

$$| \, \psi(t)\rangle \to \begin{bmatrix} \left[\cos\left(\frac{\omega_r t}{2}\right) + i\frac{\omega_0-\omega}{\omega_r}\sin\left(\frac{\omega_r t}{2}\right)\right]e^{i\omega t/2} \\ \\ \frac{i\gamma B}{\omega_r}\sin\left(\frac{\omega_r t}{2}\right)e^{-i\omega t/2} \end{bmatrix}$$

Compare this to the state $| \, \hat{\mathbf{n}}, +\rangle$ and see what is happening to the spin for the case $\omega_0 = \omega$. Show that $\langle\mu_z(t)\rangle$ is given by

$$= \langle\mu_z(0)\rangle\left[\frac{(\omega_0-\omega)^2}{(\omega_0-\omega)^2+\gamma^2B^2} + \frac{\gamma^2B^2\cos\omega_r t}{(\omega_0-\omega)^2+\gamma^2B^2}\right]$$

3. **Sakurai, p. 244, problem 15**

   The wave function of a particle subjected to a spherically symmetrical potential $V(r)$ is given by

$$\psi(\mathbf{x}) = (x + y + 3z))f(r)$$

   (a) Is ψ an eigenfunction of $\mathbf{L}^2$? If so, what is the $l-$value? If not, what are the possible values of l we may obtain when $\mathbf{L}^2$ is measured?

   (b) What are the probabilities for the particle to be found in various m_l states?

   (c) Suppose it is known somehow that $\psi(\mathbf{x})$ is an energy eigenfunction with eigenvalue E. Indicate how we may find $V(r)$.

4. **Sakurai, p. 245, problem**

   Consider an orbital angular-momentum eigenstate $\mid l = 2, m = 0\rangle$. Suppose this state is rotated by an angle β about the $y-$axis. Find the probability for the new state to be found in $m = 0, \pm 1$, and ± 2 (The spherical harmonics for $l = 0, 1$, and 2 given in Appendix A may be useful.)

5. **Sakurai, p. 245, problem 22**

   (a) Consider a system with $j = 1$. Explicity write

$$\langle j = 1, m' \mid J_y \mid j = 1, m\rangle$$

   in 3X3 matrix form.

   (b) Show that for $j = 1$ only, it is legitimate to replace $e^{-iJ_y\beta/\hbar}$ by

$$1 - \left(\frac{J_y}{\hbar}\right)\sin\beta - \left(\frac{J_y}{\hbar}\right)^2(1 - \cos\beta).$$

   (c) Using (b), prove

$$d^{(j=1)}(\beta) = \begin{pmatrix} \frac{1}{2}(1+\cos\beta) & -\frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1-\cos\beta) \\ \frac{1}{\sqrt{2}}\sin\beta & \cos\beta & -\frac{1}{\sqrt{2}}\sin\beta \\ \frac{1}{2}(1-\cos\beta) & \frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1+\cos\beta) \end{pmatrix}$$

6. **Neutrino oscillations**

   Let $\mid \nu_1\rangle$ and $\mid \nu_2\rangle$ be the neutrino mass eigenstates. They are in general different from the weak interaction electron $\mid \nu_e\rangle$, and muon $\mid \nu_\mu\rangle$ neutrino eigenstates. Electron neutrinos ν_e are produced by weak interactions in the sun. The mass eigenstates and the weak eigenstates are related according to

$$
\begin{aligned}
|\,\nu_e\rangle &= \cos\theta|\,\nu_1\rangle + \sin\theta|\,\nu_2\rangle \\
|\,\nu_\mu\rangle &= -\sin\theta|\,\nu_1\rangle + \cos\theta|\,\nu_2\rangle
\end{aligned}
$$

A nonzero mass difference between ν_1 and ν_2 would result in neutrino oscillations that would be manifested as the disappearance of solar electron neutrinos at an earth based observatory. Suppose that a ν_e neutrino is created at $t = 0$ as a momentum eigenstate. What is the probability for finding the system in state $|\,\nu_\mu\rangle$ at a later time? (This is the same formalism as problems 6& 7 from last week's assignment except there is no decay term.)