P6572 HW #8
Due October 28, 2011

1. **Isospin**
In strong interactions it is found that there is an "internal" symmetry called isospin invariance. The proton and neutron are regarded as $I_z = \frac{1}{2}$ and $I_z = -\frac{1}{2}$ states of the "nucleon". Strong interactions are invariant under rotations in isospin space, just as ordinary interactions are invariant under rotations in physical space. The generators of isospin rotations are indicated by I_x, I_y, and I_z and obey the same commutation relations as the rotation group:

$$[I_x, I_y] = iI_z, \quad \text{etc.}$$

In strong interactions, I^2 and I_z remain constant. Using the table below and the table of Clebsch Gordon coefficients, determine the relative rates for the following decays:

$$\frac{\text{Rate } \rho^0 \to \pi^0\pi^0}{\text{Rate } \rho^0 \to \pi^+\pi^-}, \quad \frac{\text{Rate } K^{*+} \to K^+\pi^0}{\text{Rate } K^{*+} \to K^0\pi^+}, \quad \frac{\text{Rate } \Delta^+ \to p\pi^0}{\text{Rate } \Delta^+ \to n\pi^+}$$

$I=1$	π^+	π^0	π^-
I_z	1	0	-1

$I=1$	ρ^+	ρ^0	ρ^-
I_z	1	0	-1

$I=\frac{1}{2}$	K^{*+}	K^{*0}
I_z	$\frac{1}{2}$	$-\frac{1}{2}$

$I=\frac{1}{2}$	K^+	K^0
I_z	$\frac{1}{2}$	$-\frac{1}{2}$

$I=\frac{3}{2}$	Δ^{++}	Δ^+	Δ^0	Δ^-
I_z	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2}$

2. **Rotations and Clebsch Gordon coefficients**
Prove

$$\mathcal{D}^j_{mm'}(R) = \sum_{m_1,m_1',m_2,m_2'} \langle j, m, j_1, j_2 \mid j_1, m_1, j_2, m_2\rangle D^{j_1}_{m_1m_1'}(R)\mathcal{D}^{j_2}_{m_2,m_2'}(R)\langle j_1, m_1', j_2, m_2' \mid j, m', j_1, j_2\rangle$$

Use this along with the known values of the rotation matrices d for $j = \frac{1}{2}$ and $j = 1$ (you calculated this last week) to compute

$$d^{\frac{3}{2}}_{\frac{3}{2},\frac{3}{2}}(\theta), \quad d^{\frac{3}{2}}_{\frac{3}{2},\frac{1}{2}}(\theta), \quad d^{\frac{3}{2}}_{\frac{1}{2},\frac{1}{2}}(\theta)$$

3. **Sakurai, p. 246, problem 25**
Consider a spherical tensor of rank 1 (that is a vector)

$$V^1_{\pm1} = \mp\frac{V_x \pm iV_y}{\sqrt{2}}, \qquad V^1_0 = V_z.$$

Using

$$d^{(j=1)}(\beta) = \begin{pmatrix} \frac{1}{2}(1+\cos\beta) & -\frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1-\cos\beta) \\ \\ \frac{1}{\sqrt{2}}\sin\beta & \cos\beta & -\frac{1}{\sqrt{2}}\sin\beta \\ \\ \frac{1}{2}(1-\cos\beta) & \frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1+\cos\beta) \end{pmatrix}$$

evaluate

$$\sum_{q'} d^1_{qq'}(\beta) V^1_{q'}$$

and show that your results are just what you expect from the transformation properties of $V_{x,y,z}$ under rotations about the $y-$ axis.

4. **Sakurai, p. 246, problem 26**

   (a) Construct a spherical tensor of rank 1 out of two different vectors $\mathbf{U} = (U_x, U_y, U_z)$ and $\mathbf{V} = (V_x, V_y, V_z)$. Explicity write $T^1_{\pm 1,0}$ in terms of $U_{x,y,z}$ and $V_{x,y,z}$.

   (b) Construct a spherical tensor of rank 2 out of two different vectors $\mathbf{U}$ and $\mathbf{V}$. Write down explicitly $T^2_{\pm 2,\pm 1,0}$ in terms of $U_{x,y,z}$ and $V_{x,y,z}$.

5. **Sakurai, p. 246, problem 27**
   Consider a spinless particle bound to a fixed center by a central force potential.

   (a) Relate, as much as possible, the matrix elements

   $$\left\langle n', l', m' \mid \mp\frac{1}{\sqrt{2}}(x \pm iy) \mid n, l, m \right\rangle \quad \text{and} \quad \langle n', l', m' \mid z \mid n, l, m\rangle$$

   using *only* the Wigner-Eckart theorem. Make sure to state under what conditions the matrix elements are nonvanishing.

   (b) Do the same problem using wave functions $\psi(\mathbf{x}) = R_{nl}(r) Y_i^m(\theta, \phi)$.

6. **Sakurai, p. 247, problem 28**

   (a) Write xy, xz, and $(x^2 - y^2)$ as components of a spherical (irreducible) tensor of rank 2.

   (b) The expectation value

   $$Q \equiv e \left\langle \alpha, j, m = j \mid (3z^2 - r^2) \mid \alpha, j, m = j \right\rangle$$

is known as the *quadrupole moment*. Evaluate

$$e \left\langle \alpha, j, m' \mid (x^2 - y^2) \mid \alpha, j, m = j \right\rangle,$$

(where $m' = j, j-1, j-2, ...$) in terms of Q and appropriate Clebsch-Gordan coefficients.

7. **Sakurai, p. 245, problem 21**

   (a) Evaluate

   $$\sum_{m=-j}^{j} |d^{(j)}_{mm'}(\beta)|^2 m$$

   for *any* j (integer or half-integer); then check your answer for $j = \frac{1}{2}$.

   (b) Prove for any j,

   $$\sum_{m=-j}^{j} m^2 |d^{(j)}_{m'm}(\beta)|^2 = \frac{1}{2} j(j+1)\sin^2\beta + m'^2 \frac{1}{2}(3\cos^2\beta - 1).$$

   [*Hint* : This can be proved in many ways. You may, for instance, examine the rotational properties of J_z^2 using the spherical (irreducible) tensor language.]

8. **Integration**
   Show that

$$\int_0^{4\pi} \frac{d\alpha}{4\pi} \int_0^{4\pi} \frac{d\gamma}{4\pi} \int_0^{\pi} \frac{\sin\beta d\beta}{2} \mathcal{D}^j_{m',m}(\alpha, \beta, \gamma) = \delta_{m',0}\delta_{m,0}\delta_{j,0}$$

where

$$\mathcal{D}^j_{m'm}(\alpha, \beta, \gamma) = \left\langle jm' \mid e^{-\frac{i}{\hbar}J_z\alpha} e^{-\frac{i}{\hbar}J_y\beta} e^{-\frac{i}{\hbar}J_z\gamma} \mid jm \right\rangle$$

It might be helpful to remember that

$$D^l_{m,0}(\alpha, \beta, 0) = \sqrt{\frac{4\pi}{2l+1}} Y_l^{m^*}(\theta, \phi)|_{\theta=\beta, \phi=\alpha}$$

and that

$$L^2 Y_l^m = \hbar^2 l(l+1) Y_l^m$$