# Problem Set 8 – Solutions

## Problem 1

The decays in question will be given by some Hadronic matric element:

$$\Gamma \propto |\langle i|V|f\rangle|^2$$

where $|i\rangle$ is the initial state, $V$ is an interaction term, and $|f\rangle$ is the final state. The strong interaction preserves isospin, so $V$ must be an isospin singlet. We apply the Wigner-Eckart theorem to compute the required ratios:

$$\langle i|V|f\rangle \;\propto\; \langle i|f\rangle\,\langle i\|V\|f\rangle$$

where $\langle i\|V\|f\rangle$ is an isospin singlet which cancels upon taking ratios of two matrix elements which differ by an isospin rotation. Thus, we can compute ratios of certain rates by taking ratios of Clebsch-Gordan coefficients.

Consider the first ratio:

$$\frac{\Gamma(\rho^0 \to \pi^0\,\pi^0)}{\Gamma(\rho^0 \to \pi^+\,\pi^-)}$$

We have:

$$|1,0\rangle \;=\; \frac{1}{\sqrt{2}}\,(|1,1\rangle \otimes |1,-1\rangle - |1,-1\rangle \otimes |1,1\rangle\Big)$$

Thus, $\langle 1,0|(|1,0\rangle \otimes |1,0\rangle) = 0$ and the numerator vanishes, so

$$\frac{\Gamma(\rho^0 \to \pi^0\,\pi^0)}{\Gamma(\rho^0 \to \pi^+\,\pi^-)} = 0$$

Now consider the second ratio

$$\frac{\Gamma(K^{\star +} \to K^+\,\pi^0)}{\Gamma(K^{\star +} \to K^0\,\pi^+)} \;=\; \frac{|\left\langle \frac{1}{2},\frac{1}{2}\right|\Big(\left|\frac{1}{2},\frac{1}{2}\right\rangle \otimes |1,0\rangle\Big)|^2}{|\left\langle \frac{1}{2},\frac{1}{2}\right|\Big(\left|\frac{1}{2},-\frac{1}{2}\right\rangle \otimes |1,1\rangle\Big)|^2} \;=\; \frac{|\sqrt{1/3}|^2}{|\sqrt{2/3}|^2} \;=\; \frac{1}{2}$$

Finally, the third ratio is:

$$\frac{\Gamma(\Delta^+ \to p\,\pi^0)}{\Gamma(\Delta^+ \to n\,\pi^+)} \;=\; \frac{|\left\langle \frac{3}{2},\frac{1}{2}\right|\Big(\left|\frac{1}{2},\frac{1}{2}\right\rangle \otimes |1,0\rangle\Big)|^2}{|\left\langle \frac{3}{2},\frac{1}{2}\right|\Big(\left|\frac{1}{2},-\frac{1}{2}\right\rangle \otimes |1,1\rangle\Big)|^2} \;=\; \frac{|\sqrt{2/3}|^2}{|\,\sqrt{1/3}|^2} \;=\; 2$$

In reality, isopin symmetry is broken by a number of effects, including electromagnetic interactions (e.g. the proton and the neutron have different charges), the weak interaction, and the light-quark masses. However, for decays mediated by the strong force, isospin is a good symmetry, and should be approximately conserved. For other decays, isospin is *not* conserved. For instance, isospin conservation would suggest that all the pions are stable with equal masses, since they form an isospin triplet and all lighter particles are isospin singlets (apart from bare quarks). However, the $\pi^{\pm}$ are in fact $\sim 5$ MeV *heavier* than the $\pi^0$, and all pions are unstable. The $\pi^0$ has a lifetime of $\sim 10^{-16}\,s$ and decays electromagnetically, whereas the $\pi^{\pm}$ have equal lifetimes of $\sim 10^{-8}\,s$ and decay via the weak interaction.

# Problem 2

The matrix elements $\mathcal{D}^{(j)}_{m'm}(R)$ are defined by $\langle j,\, m'|\mathcal{D}(R)|j,\, m\rangle$ where $\mathcal{D}(R)$ is the operator associated with the spatial rotation $R$. Now consider the direct product state $|j_1\ j_2;\ m_1\ m_2\rangle = |j_1,\, m_1\rangle \otimes |j_2,\, m_2\rangle$. $\mathcal{D}(R)$ rotates each factor equally:

$$\mathcal{D}(R)\,|j_1\,j_2;m_1\,m_2\rangle \;=\; [\mathcal{D}(R)\,|j_1,m_1\rangle]\otimes[\mathcal{D}(R)\,|j_2,m_2\rangle\,]$$

Therefore,

$$\langle j_1\,j_2;m_1\,m_2|\,\mathcal{D}(R)\,|j_1\,j_2;m_1'\,m_2'\rangle \;=\; \langle j_1,m_1|\mathcal{D}(R)\,|j_1,m_1'\rangle\,\langle j_2,m_2|\mathcal{D}(R)\,|j_2,m_2'\rangle \;=\; \mathcal{D}^{j_1}_{m_1\,m_1'}(R)\,\mathcal{D}^{j_2}_{m_2\,m_2'}(R)$$

and so

$$\begin{aligned}
\mathcal{D}^{(j)}_{m\,m'} &= \langle j,m|\mathcal{D}(R)|j,m'\rangle\\
&= \langle j_1\,j_2;\,j\,m|\mathcal{D}(R)|j_1\,j_2;\,j\,m'\rangle\\
&= \sum_{m_1,m_2}\sum_{m_1'\,m_2'}\langle j_1\,j_2;\,j\,m|j_1j_2;m_1\,m_2\rangle\langle j_1j_2;m_1\,m_2|\mathcal{D}(R)|j_1j_2;m_1'\,m_2'\rangle\langle j_1j_2;m_1'\,m_2'|j_1\,j_2;\,j\,m\rangle\\
&= \sum_{m_1,m_2}\sum_{m_1'\,m_2'}\langle j_1\,j_2;\,j\,m|j_1j_2;m_1\,m_2\rangle\,\mathcal{D}^{j_1}_{m_1\,m_1'}(R)\,\mathcal{D}^{j_2}_{m_2\,m_2'}(R)\,\langle j_1j_2;m_1'\,m_2'|j_1\,j_2;\,j\,m\rangle
\end{aligned}$$

Since $d^{(j)}_{m\,m'}(\theta)=\mathcal{D}^{(j)}_{m\,m'}(R(\theta))$, where $R(\theta)$ is a rotation about the $y$-axis by an angle $\theta$, we have

$$d^{(j)}_{m\,m'}(\theta) \;=\; \sum_{m_1,m_2}\sum_{m_1'\,m_2'}\langle j_1\,j_2;\,j\,m|j_1j_2;m_1\,m_2\rangle\,d^{j_1}_{m_1\,m_1'}(\theta)\,d^{j_2}_{m_2\,m_2'}(\theta)\,\langle j_1j_2;m_1'\,m_2'|j_1\,j_2;\,j\,m\rangle$$

We are free to choose any $j_1, j_2$ which satisfy $j_1+j_2=j$. Thus, to find the spin-3/2 matrices, we take $j_1=1$ and $j_2=1/2$. We find:

$$\begin{aligned}
d^{3/2}_{3/2,3/2}(\theta) &= \left\langle \frac{3}{2},\frac{3}{2}\middle|1,\frac{1}{2};1,\frac{1}{2}\right\rangle d^{1}_{1,1}(\theta)\,d^{1/2}_{1/2,1/2}(\theta)\left\langle 1,\frac{1}{2};1,\frac{1}{2}\middle|\frac{3}{2},\frac{3}{2}\right\rangle\\
&= d^{1}_{1,1}(\theta)\,d^{1/2}_{1/2,1/2}(\theta) \;=\; \frac{1}{2}\,(1+\cos\theta)\cos\frac{\theta}{2}\\
&= \cos^3\frac{\theta}{2}
\end{aligned}$$

and

$$\begin{aligned}
d^{3/2}_{3/2,1/2}(\theta) &= \left\langle \frac{3}{2},\frac{3}{2}\middle|1,\frac{1}{2};1,\frac{1}{2}\right\rangle d^{1}_{1,1}(\theta)\,d^{1/2}_{1/2,-1/2}(\theta)\left\langle 1,\frac{1}{2};1,-\frac{1}{2}\middle|\frac{3}{2},\frac{1}{2}\right\rangle\\
&\quad+\left\langle \frac{3}{2},\frac{3}{2}\middle|1,\frac{1}{2};1,\frac{1}{2}\right\rangle d^{1}_{1,0}(\theta)\,d^{1/2}_{1/2,1/2}(\theta)\left\langle 1,\frac{1}{2};0,\frac{1}{2}\middle|\frac{3}{2},\frac{1}{2}\right\rangle\\
&= \sqrt{\frac{1}{3}}\,d^{1}_{1,1}(\theta)\,d^{1/2}_{1/2,-1/2}(\theta)+\sqrt{\frac{2}{3}}\,d^{1}_{1,0}(\theta)\,d^{1/2}_{1/2,1/2}(\theta)\\
&= -\sqrt{\frac{1}{3}}\,\frac{1}{2}\,(1+\cos\theta)\sin\frac{\theta}{2}-\sqrt{\frac{1}{3}}\sin\theta\,\cos\frac{\theta}{2}\\
&= -\sqrt{3}\,\sin\frac{\theta}{2}\,\cos^2\frac{\theta}{2}
\end{aligned}$$

and

$$
\begin{aligned}
d^{3/2}_{1/2,1/2}(\theta) &= \left\langle \frac{3}{2},\frac{1}{2}\right|1,\frac{1}{2};1,-\frac{1}{2}\Big\rangle\, d^{1}_{1,1}(\theta)\; d^{1/2}_{-1/2,-1/2}(\theta)\,\Big\langle 1,\frac{1}{2};1,-\frac{1}{2}\left|\frac{3}{2},\frac{1}{2}\right\rangle \\
&+ \left\langle \frac{3}{2},\frac{1}{2}\right|1,\frac{1}{2};1,-\frac{1}{2}\Big\rangle\, d^{1}_{1,0}(\theta)\; d^{1/2}_{-1/2,1/2}(\theta)\,\Big\langle 1,\frac{1}{2};0,\frac{1}{2}\left|\frac{3}{2},\frac{1}{2}\right\rangle \\
&+ \left\langle \frac{3}{2},\frac{1}{2}\right|1,\frac{1}{2};0,\frac{1}{2}\Big\rangle\, d^{1}_{0,1}(\theta)\; d^{1/2}_{1/2,-1/2}(\theta)\,\Big\langle 1,\frac{1}{2};1,-\frac{1}{2}\left|\frac{3}{2},\frac{1}{2}\right\rangle \\
&+ \left\langle \frac{3}{2},\frac{1}{2}\right|1,\frac{1}{2};0,\frac{1}{2}\Big\rangle\, d^{1}_{0,0}(\theta)\; d^{1/2}_{1/2,1/2}(\theta)\,\Big\langle 1,\frac{1}{2};0,\frac{1}{2}\left|\frac{3}{2},\frac{1}{2}\right\rangle \\
&= \frac{1}{3}\, d^{1}_{1,1}(\theta)\; d^{1/2}_{-1/2,-1/2}(\theta) + \frac{\sqrt{2}}{3}\left( d^{1}_{1,0}(\theta)\; d^{1/2}_{-1/2,1/2}(\theta) + d^{1}_{0,1}(\theta)\; d^{1/2}_{1/2,-1/2}(\theta)\right) \\
&+ \frac{2}{3}\, d^{1}_{0,0}(\theta)\; d^{1/2}_{1/2,1/2}(\theta) \\
&= \frac{1}{3}\cos^3\frac{\theta}{2} - \frac{2}{3}\sin\theta\,\sin\frac{\theta}{2} + \frac{2}{3}\cos\theta\,\cos\frac{\theta}{2} \\
&= \frac{1}{3}\cos^3\frac{\theta}{2} + \frac{2}{3}\cos\frac{3\theta}{2}
\end{aligned}
$$

The other components of $d^{3/2}_{m\,m'}(\theta)$ can be worked out in a similar fashion.

## Problem 3

We can represent $V^1_q$ as a column vector:

$$
V^1_q = \begin{pmatrix} V^1_1 \\ V^1_0 \\ V^1_{-1} \end{pmatrix} = \begin{pmatrix} -\frac{V_x + i\,V_y}{\sqrt{2}} \\ V_z \\ \frac{V_x - i\,V_y}{\sqrt{2}} \end{pmatrix}
$$

Thus, $\tilde{V}^1_q \equiv \sum_{q'}\; d^1_{q\,q'}(\beta)\, V^1_{q'}$ is given by the matrix equation:

$$
\begin{aligned}
\tilde{V}^1_q &= \begin{pmatrix} \frac{1+\cos\beta}{2} & -\frac{1}{\sqrt{2}}\sin\beta & \frac{1-\cos\beta}{2} \\ \frac{1}{\sqrt{2}}\sin\beta & \cos\beta & -\frac{1}{\sqrt{2}}\sin\beta \\ \frac{1-\cos\beta}{2} & \frac{1}{\sqrt{2}}\sin\beta & \frac{1+\cos\beta}{2} \end{pmatrix} \begin{pmatrix} -\frac{V_x + i\,V_y}{\sqrt{2}} \\ V_z \\ \frac{V_x - i\,V_y}{\sqrt{2}} \end{pmatrix} \\
&= \begin{pmatrix} -\frac{i\,V_y}{\sqrt{2}} - \frac{V_x\cos\beta}{\sqrt{2}} - \frac{V_z\sin\beta}{\sqrt{2}} \\ -V_x\sin\beta + V_z\cos\beta \\ -\frac{i\,V_y}{\sqrt{2}} + \frac{V_x\cos\beta}{\sqrt{2}} - \frac{V_z\sin\beta}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -\frac{(V_x\cos\beta + V_z\sin\beta) + i\,V_y}{\sqrt{2}} \\ V_z\cos\beta - V_x\sin\beta \\ \frac{(V_x\cos\beta + V_z\sin\beta) - i\,V_y}{\sqrt{2}} \end{pmatrix}
\end{aligned}
$$

Therefore, $\tilde{V}_x = V_x\cos\beta + V_z\sin\beta$ and $\tilde{V}_z = V_z\cos\beta - V_x\sin$ <nonesep> $\beta$

$$
\begin{pmatrix} \tilde{V}_x \\ \tilde{V}_y \\ \tilde{V}_z \end{pmatrix} = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}
$$

This is clearly a rotation about the $y$ axis by an angle $\beta$; one can even check that the signs are correct using the right-hand rule.

# Problem 4

This problem is a straightforward application of (3.11.26) in Sakurai.

## a)

Given vectors $\boldsymbol{U} = (U_x, U_y, U_z)$ and $\boldsymbol{V} = (V_x, V_y, V_z)$, we construct the usual cross product:

$$\boldsymbol{W} \;=\; \boldsymbol{U}\times\boldsymbol{V} \;=\; (U_y V_z - U_z V_y,\; U_z V_x - U_x V_z,\; U_x V_y - U_y V_x)$$

We construct a spherical tensor out of $\boldsymbol{W}$ by the standard procedure, as in the previous problem:

$$W_{\pm 1} \;=\; \mp\frac{W_x \pm i W_y}{\sqrt{2}} \quad , \quad W_0 \;=\; W_z$$

We then have:

$$T_q^{(1)} \;=\; \frac{(\boldsymbol{U}\times\boldsymbol{V})_q}{i\sqrt{2}} \;=\; \frac{W_q}{i\sqrt{2}} \;=\; \begin{pmatrix} -\frac{U_y V_z - U_z V_y + i\,(U_z V_x - U_x V_z)}{2i} \\ \frac{U_x V_y - U_y V_x}{i\sqrt{2}} \\ \frac{U_y V_z - U_z V_y - i\,(U_z V_x - U_x V_z)}{2i} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}\left[V_z\,(U_x + iU_y) - U_z\,(V_x + iV_y)\right] \\ \frac{1}{i\sqrt{2}}\,(U_x V_y - U_y V_x) \\ \frac{1}{2}\left[V_z\,(U_x - iU_y) - U_z\,(V_x - iV_y)\right] \end{pmatrix}$$

where the overall normalization is merely conventional.

## c)

From Saurai (3.11.26), we find:

$$T_{\pm 2}^{(2)} \;=\; U_{\pm 1} V_{\pm 1} \;=\; \frac{1}{2}(U_x \pm iU_y)(V_x \pm iV_y) \;=\; \frac{1}{2}(U_x V_x - U_y V_y) \pm \frac{i}{2}(U_x V_y + U_y V_x)$$

and

$$T_{\pm 1}^{(2)} \;=\; \frac{U_{\pm 1} V_0 + U_0 V_{\pm 1}}{\sqrt{2}} \;=\; \mp\frac{1}{2}\left[V_z\,(U_x \pm iU_y) + U_z\,(V_x \pm iV_y)\right] \;=\; \mp\frac{1}{2}\left[V_z U_x + U_z V_x\right] - \frac{i}{2}\left[V_z U_y + U_z V_y\right]$$

and

$$T_0^{(2)} \;=\; \frac{U_{+1} V_{-1} + 2\,U_0 V_0 + U_{-1} V_{+1}}{\sqrt{6}}$$

$$= \frac{2\,U_z V_z - (U_x + iU_y)(V_x - iV_y)/2 - (U_x - iU_y)(V_x + iV_y)/2}{\sqrt{6}}$$

$$= \frac{2\,U_z V_z - U_x V_x - U_y V_y}{\sqrt{6}}$$

We recognize various components of the symmetric/traceless piece of $T_{ij} \equiv U_i V_i$.

# Problem 5

## a)

We define the spherical tensor:

$$X^1_{\pm 1} \;=\; \mp \frac{x \pm i y}{\sqrt{2}} \quad , \quad X^1_0 \;=\; z$$

Thus, the matrix elements we are asked to related are those of $X^1_{\pm 1, 0}$. We apply the Wigner-Eckart theorem, (3.11.31), to obtain:

$$\big\langle n', l', m' | X^1_q | n, l, m \big\rangle \;=\; \langle l, 1; m, q | l', m' \rangle \, \mathcal{M}_X(n, n', l, l')$$

where $\mathcal{M}_X(n, n', l, l')$ is the rotationally invariant piece of the matrix element, independent of $m, m'$ and $q$. The above expression relates the matrix elements of $X^1_q$ for different $q$. We see that the matrix element vanishes unless:

$$m' = m + q \quad , \quad |l - 1| \leqslant l' \leqslant |l + 1|$$

since the Clebsch-Gordon coefficient vanishes unless these conditions are satisfied by conservation of angular momentum. In certain other special cases the Clebsch-Gordan coefficient will vanish even when these conditions are obeyed, e.g. for $l = l' = 1$ and $m = m' = q = 0$.

## b)

Instead of applying the Wigner-Eckart theorem, we rewrite the matrix element in the position basis by standard methods:

$$\begin{aligned}
\big\langle n', l', m' | X^1_q | n, l, m \big\rangle \;&=\; \int \mathrm{d}^3 x \; \langle n', l', m' | X^1_q | \boldsymbol{x} \rangle \langle \boldsymbol{x} | n, l, m \rangle \\
&=\; \int \mathrm{d}^3 x \; \langle n', l', m' | \boldsymbol{x} \rangle \langle \boldsymbol{x} | n, l, m \rangle \, X^1_q(x) \\
&=\; \int \mathrm{d}^3 x \, \Psi_{n', l', m'}(\boldsymbol{x})^\star \, \Psi_{n, l, m}(\boldsymbol{x}) \, X^1_q(x)
\end{aligned}$$

The wavefunction $\Psi_{n,l,m}(\boldsymbol{x})$ should take the form

$$\Psi_{n,l,m}(\boldsymbol{x}) \;=\; Y^m_l(\theta, \phi) \, R_{n\,l}(r)$$

for some unknown radial wavefunction $R_{n\,l}(r)$. Moreover,

$$X^1_q(x) \;=\; \begin{pmatrix} -\frac{x + i y}{\sqrt{2}} \\ z \\ \frac{x - i y}{\sqrt{2}} \end{pmatrix} \;=\; \begin{pmatrix} -\frac{1}{\sqrt{2}} r \sin\theta \, e^{i\phi} \\ r \cos\theta \\ \frac{1}{\sqrt{2}} r \sin\theta \, e^{-i\phi} \end{pmatrix} \;=\; \sqrt{\frac{4\pi}{3}} \, r \, Y^q_1(\theta, \phi)$$

Thus, since $\mathrm{d}^3 x = r^2 \, \mathrm{d}r \, \mathrm{d}\Omega$, we find

$$\big\langle n', l', m' | X^1_q | n, l, m \big\rangle \;=\; \left[ \sqrt{\frac{4\pi}{3}} \int R_{n'\,l'}(r)^\star \, R_{n\,l}(r) \, r^3 \, \mathrm{d}r \right] \times \int \mathrm{d}\Omega \, Y^{m'}_{l'}(\theta, \phi)^\star \, Y^m_l(\theta, \phi) \, Y^q_1(\theta, \phi)$$

We now apply (3.8.73) from Sakurai, namely

$$\int d\Omega \, Y_l^m(\theta,\phi)^\star \, Y_{l_1}^{m_1}(\theta,\phi) \, Y_{l_2}^{m_2}(\theta,\phi) \;=\; \mathcal{N}_{l;\,l_1,l_2} \, \langle l_1 \, l_2; m_1 \, m_2 | l_1 \, l_2; l \, m \rangle$$

where $\mathcal{N}_{l,l_1,l_2}$ is a rotationally invariant numerical factor which we do not write. Thus,

$$\big\langle n', l', m' | X_q^1 | n, l, m \big\rangle \;=\; \left[ \sqrt{\frac{4\pi}{3}} \, \mathcal{N}_{l';\,l,1} \int R_{n'\,l'}(r)^\star \, R_{nl}(r) \, r^3 \, dr \right] \times \langle \, l, 1; m, q \, | l' \, m' \rangle$$

and we recover our result from part (a), where

$$\mathcal{M}_X(n,n',l,l') \;=\; \sqrt{\frac{4\pi}{3}} \, \mathcal{N}_{l';\,l,1} \int R_{n'\,l'}(r)^\star \, R_{nl}(r) \, r^3 \, dr$$

and the exact form of $\mathcal{N}_{l';\,l,1}$ is given by Sakurai (3.8.73).

# Problem 6

## a)

This is a simple application of problem 4, with $\boldsymbol{U} = \boldsymbol{V} = \boldsymbol{x}$. We find:

$$\begin{aligned}
Q^{(2)}_{\pm 2} \;&=\; X^2_{\pm 1} \;=\; \frac{1}{2}\,(x \pm iy)^2 \;=\; \frac{1}{2}\,(x^2 - y^2) \pm i\,x\,y \\
Q^{(2)}_{\pm 1} \;&=\; \sqrt{2}\,X_{\pm 1}\,X_0 \;=\; \mp\,z\,(x \pm iy) \;=\; \mp\,x\,z - i\,y\,z \\
Q^{(2)}_{0} \;&=\; \frac{2\,X_0\,X_0 + 2\,X_{+1}\,X_{-1}}{\sqrt{6}} \;=\; \frac{2\,z^2 - (x+iy)(x-iy)}{\sqrt{6}} \;=\; \sqrt{\frac{1}{6}}\,\big(2\,z^2 - x^2 - y^2\big)
\end{aligned}$$

Therefore,

$$\begin{aligned}
x^2 - y^2 \;&=\; Q_{+2} + Q_{-2} \\
xy \;&=\; \frac{1}{2\,i}\,[Q_{+2} - Q_{-2}] \\
xz \;&=\; -\frac{1}{2}\,[Q_{+1} - Q_{-1}] \\
yz \;&=\; -\frac{1}{2i}\,[Q_{+1} + Q_{-1}] \\
2\,z^2 - x^2 - y^2 \;&=\; \sqrt{6}\,Q_0
\end{aligned}$$

## b)

We have:

$$\begin{aligned}
Q \;&=\; e\,\langle \alpha, j, m = j\,|(3\,z^2 - r^2)|\alpha, j, m = j \rangle \;=\; \sqrt{6}\,e\,\langle \alpha, j, m = j\,|Q_0|\alpha, j, m = j \rangle \\
&=\; \sqrt{6}\,e\,\langle j, 2; j, 0 | j, j \rangle \; \mathcal{M}_Q(\alpha, j, \alpha, j)
\end{aligned}$$

and

$$
\begin{aligned}
e\,\langle\alpha,j,m'|\,x^2-y^2\,|\alpha,j,m\!=\!j\rangle &= e\,\langle\alpha,j,m'|Q_{+2}+Q_{-2}\,|\alpha,j,m\!=\!j\rangle \\
&= e\,\langle\alpha,j,m'|Q_{-2}\,|\alpha,j,m\!=\!j\rangle \\
&= e\,\langle j,2;j,-2|j,m'\rangle\;\mathcal{M}_Q(\alpha,j,\alpha,j) \\
&= \frac{\langle j,2;j,-2|j,m'\rangle}{\sqrt{6}\,\langle j,2;j,0|j,j\rangle}\,Q
\end{aligned}
$$

where the $Q_{+2}$ term vanishes, since $m' = j + 2$ is impossible. We readily see that the matrix element is only nonvanishing for $m' = j-2$, in which case:

$$
e\,\langle\alpha,j,m'\!=\!j-2|\,x^2-y^2\,|\alpha,j,m\!=\!j\rangle \;=\; \frac{\langle j,2;j,-2|j,j-2\rangle}{\sqrt{6}\,\langle j,2;j,0|j,j\rangle}\,Q
$$

# Problem 7

## a)

We find

$$
\begin{aligned}
\sum_{m=-j}^{j} m\left|d^j_{mm'}(\beta)\right|^2 &= \sum_{m=-j}^{j}\langle j,m'|e^{\frac{i}{\hbar}J_y\beta}m\,|j,m\rangle\langle j,m|e^{-\frac{i}{\hbar}J_y\beta}|j,m'\rangle \\
&= \sum_{j''}\sum_{m=-j''}^{j''}\langle j,m'|e^{\frac{i}{\hbar}J_y\beta}m|j'',m\rangle\langle j'',m|e^{-\frac{i}{\hbar}J_y\beta}|j,m'\rangle \\
&= \frac{1}{\hbar}\langle j,m'|e^{\frac{i}{\hbar}J_y\beta}\,J_z\,e^{-\frac{i}{\hbar}J_y\beta}|j,m'\rangle
\end{aligned}
$$

where $d^j_{mm'}(\beta)$ is defined as

$$
d^j_{mm'}(\beta) \;\equiv\; \langle j,m|e^{-\frac{i}{\hbar}J_y\beta}|j,m'\rangle
$$

and we use the fact that $\langle j,m'|e^{-\frac{i}{\hbar}J_y\beta}|j'',m\rangle$ vanishes for $j\neq j''$, together with the spectral decomposition of $J_z$:

$$
J_z \;=\; \sum_{j=0,\frac{1}{2},1,\ldots}\;\sum_{m=-j}^{j}\hbar m|j,m\rangle\langle j,m|
$$

To interpret the resulting expression, recall that $\boldsymbol{J}$ is the generator of rotations, in that

$$
\mathcal{U}(\hat{\boldsymbol{n}},\theta) \;\equiv\; e^{-\frac{i}{\hbar}\theta\,(\boldsymbol{J}\cdot\hat{\boldsymbol{n}})}
$$

represents a rotation by an angle $\theta$ about the axis $\hat{\boldsymbol{n}}$, with direction of rotation determined by the right-hand rule, e.g.

$$
\mathcal{U}(\hat{y},\beta)|\boldsymbol{x}_0\rangle \;=\; \left|\boldsymbol{x}_0'\right\rangle \;\;,\;\; \boldsymbol{x}_0' = (x_0\cos\beta+z_0\sin\beta)\,\hat{x}+y_0\,\hat{y}+(z_0\cos\beta-x_0\sin\beta)\,\hat{z}
$$

Applying the operator $\boldsymbol{x}$ to both sides, we see that $\mathcal{U}$ can also be interpreted as rotating the operator $\boldsymbol{x}$:

$$\boldsymbol{x}' \;=\; \mathcal{U}(\hat{y},\beta)^\dagger\, \boldsymbol{x}\; \mathcal{U}(\hat{y},\beta) \quad , \quad \boldsymbol{x}' = (x\cos\beta + z\sin\beta)\,\hat{x} + y\,\hat{y} + (z\cos\beta - x\sin\beta)\,\hat{z}$$

(These two viewpoints are similar to the "active" and "passive" interpretations of rotations, respectively, and are analogous to the Schrödinger and Heisenberg pictures for time translation.)

This result generalizes to other vector operators. Thus,

$$\mathcal{U}(\hat{y},\beta)^\dagger\, J_z\, \mathcal{U}(\hat{y},\beta) \;=\; e^{\frac{i}{\hbar}J_y\beta}\, J_z\, e^{-\frac{i}{\hbar}\beta J_y} \;=\; J_z\cos\beta - J_x\sin\beta$$

Alternately, we could have established this identity by brute force, e.g. by applying the Cambell-Baker-Hausdorff equation.

Thus,

$$\sum_{m=-j}^{j} m\left|d^j_{m\,m'}(\beta)\right|^2 \;=\; \frac{1}{\hbar}\,\langle j,m'|\,(J_z\cos\beta - J_x\sin\beta)\,|j,m'\rangle \;=\; m'\cos\beta$$

since $\langle j,m|J_x|j,m'\rangle$ is only nonvanishing for $m' = m\pm 1$.

For the special case $j = 1/2$, we have

$$d^{1/2}_{mm'} \;=\; \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}$$

Thus,

$$\sum_{m=-\frac{1}{2},\frac{1}{2}} m\left|d^{1/2}_{m,1/2}(\beta)\right|^2 \;=\; -\frac{1}{2}\sin^2\frac{\beta}{2} + \frac{1}{2}\cos^2\frac{\beta}{2} \;=\; \frac{1}{2}\cos\beta$$

$$\sum_{m=-\frac{1}{2},\frac{1}{2}} m\left|d^{1/2}_{m,-1/2}(\beta)\right|^2 \;=\; -\frac{1}{2}\cos^2\frac{\beta}{2} + \frac{1}{2}\sin^2\frac{\beta}{2} \;=\; -\frac{1}{2}\cos\beta$$

as expected.

## b)

We have

$$\begin{aligned}
\sum_{m=-j}^{j} m^2\left|d^j_{m'\,m}(\beta)\right|^2 \;&=\; \sum_{m=-j}^{j} \langle j,m'|e^{-\frac{i}{\hbar}J_y\beta} m^2\,|j,m\rangle\langle j,m|e^{\frac{i}{\hbar}J_y\beta}|j,m'\rangle \\
&=\; \sum_{j''}\sum_{m=-j''}^{j''} \langle j,m'|e^{-\frac{i}{\hbar}J_y\beta} m^2\,|j'',m\rangle\langle j'',m|e^{\frac{i}{\hbar}J_y\beta}|j,m'\rangle \\
&=\; \frac{1}{\hbar^2}\,\langle j,m'|e^{-\frac{i}{\hbar}J_y\beta} J_z^2\, e^{\frac{i}{\hbar}J_y\beta}|j,m'\rangle
\end{aligned}$$

similar to before. Note that

$$e^{-\frac{i}{\hbar}J_y\beta} J_z^2\, e^{\frac{i}{\hbar}J_y\beta} \;=\; \left(e^{-\frac{i}{\hbar}J_y\beta} J_z\, e^{\frac{i}{\hbar}J_y\beta}\right)^2$$

and

$$e^{-\frac{i}{\hbar}J_y\beta}J_z\,e^{\frac{i}{\hbar}J_y\beta} \;=\; J_z\cos\beta + J_x\sin\beta$$

similar to what we found in part $(a)$. Thus,

$$e^{-\frac{i}{\hbar}J_y\beta}J_z^2\,e^{\frac{i}{\hbar}J_y\beta} \;=\; J_z^2\cos^2\,\beta + (J_x\,J_z + J_z\,J_x)\cos\beta\sin\beta + J_x^2\sin^2\,\beta$$

We then find

$$\sum_{m=-j}^{j} m^2\left|d^j_{m'\,m}(\beta)\right|^2 \;=\; m'^2\cos^2\,\beta + \frac{1}{\hbar^2}\sin^2\,\beta\,\langle j,m'|J_x^2|j,m'\rangle$$

since $\langle j,m'|J_x|j,m'\rangle = 0$ as before. To evaluate the last term, we write $J_x$ in terms of ladder operators:

$$J_x \;=\; \frac{1}{2}(J_+ + J_-)$$

Thus,

$$\langle j,m'|J_x^2|j,m'\rangle \;=\; \frac{1}{4}\langle j,m'|J_+\,J_- + J_-\,J_+|j,m'\rangle$$

since $\langle j,m'|J_+^2|j,m'\rangle = \langle j,m'|J_-^2|j,m'\rangle = 0$. However,

$$J_+\,J_- + J_-\,J_+ \;=\; (J_x + i\,J_y)(J_x - i\,J_y) + (J_x - i\,J_y)(J_x + i\,J_y) \;=\; 2\left(J_x^2 + J_y^2\right) \;=\; 2\left(J^2 - J_z^2\right)$$

Thus,

$$\frac{1}{\hbar^2}\langle j,m'|J_x^2|j,m'\rangle \;=\; \frac{1}{2}\left[\,j\,(j+1) - m'^2\right]$$

and so

$$\sum_{m=-j}^{j} m^2\left|d^j_{m'\,m}(\beta)\right|^2 \;=\; m'^2\cos^2\,\beta + \frac{1}{2}\sin^2\,\beta\left[\,j\,(j+1) - m'^2\right] \;=\; \frac{j\,(j+1)\sin^2\,\beta + m'^2\left(3\cos^2\,\beta - 1\right)}{2}$$

which is the desired result.

# Problem 8

We have

$$\mathcal{D}^j{}_{m'm}(\alpha,\beta,\gamma) \;\equiv\; \langle j,m'|e^{-\frac{i}{\hbar}J_z\alpha}\,e^{-\frac{i}{\hbar}J_y\beta}\,e^{-\frac{i}{\hbar}J_z\gamma}|j,m\rangle \;=\; e^{-i\,(m'\,\alpha+m\gamma)/\hbar}\,d^j_{mm'}(\beta)$$

where

$$d^j_{m'\,m}(\beta) \;\equiv\; \langle j,m'|e^{-\frac{i}{\hbar}J_y\beta}|j,m\rangle$$

Thus, since

$$\int_0^{4\pi}\frac{\mathrm{d}\alpha}{4\pi}\int_0^{4\pi}\frac{\mathrm{d}\gamma}{4\pi}\,e^{-i\,(m'\,\alpha+m\gamma)/\hbar} \;=\; \delta_{m',0}\,\delta_{m,0}$$

we obtain

$$\int_0^{4\pi} \frac{\mathrm{d}\alpha}{4\pi} \int_0^{4\pi} \frac{\mathrm{d}\gamma}{4\pi} \int_0^{\pi} \frac{\sin\beta\,\mathrm{d}\beta}{2} \mathcal{D}^j{}_{m'm}(\alpha,\beta,\gamma) \;=\; \delta_{m',0}\,\delta_{m,0} \int_{-1}^{1} \frac{\mathrm{d}\,(\cos\beta)}{2}\, d^j_{m'm}(\beta)$$

However,

$$d^j_{00}(\beta) \;=\; P_j\,(\cos\beta)$$

where $P_j(x)$ is the $j$th Legendre polynomial ($j$ cannot be half-integral, since $m = m' = 0$.) Thus,

$$\int_0^{4\pi} \frac{\mathrm{d}\alpha}{4\pi} \int_0^{4\pi} \frac{\mathrm{d}\gamma}{4\pi} \int_0^{\pi} \frac{\sin\beta\,\mathrm{d}\beta}{2} \mathcal{D}^j{}_{m'm}(\alpha,\beta,\gamma) \;=\; \delta_{m',0}\,\delta_{m,0} \int_{-1}^{1} \frac{\mathrm{d}x}{2}\, P_j(x)\, P_0(x)$$

where we insert $P_0(x) = 1$. However, the Legendre polynomials obey the orthogonality condition:

$$\int_{-1}^{1} \frac{\mathrm{d}x}{2}\, P_j(x)\, P_{j'}(x) \;=\; \frac{1}{2j+1}\,\delta_{jj'}$$

Thus,

$$\int_0^{4\pi} \frac{\mathrm{d}\alpha}{4\pi} \int_0^{4\pi} \frac{\mathrm{d}\gamma}{4\pi} \int_0^{\pi} \frac{\sin\beta\,\mathrm{d}\beta}{2} \mathcal{D}^j{}_{m'm}(\alpha,\beta,\gamma) \;=\; \delta_{m',0}\,\delta_{m,0}\,\delta_{j,0}$$

as desired.