P6572 HW #9
Due November 4, 2011

1. **Deuteron**
   The deuteron (d) is an atomic nucleus with spin $J = 1$ that is composed of a proton (p) and a neutron (n) bound together. (Both the p and the n have spin 1/2). The p and n couple with the orbital angular momentum l to make the total angular momentum (spin) j of the deuteron.

   (a) What are the possible spin and orbital angular momenta of the proton and neutron in the deuteron? Use the standard spectroscopic notation, ${}^{2S+1}L_J$.

   (b) Using tables of Clebsch-Gordon coefficients, write each of these deuteron spin states in terms of the appropriate orbital states $\mid L, M_L\rangle$ and appropriate singlet and triplet states, $\mid S, M_s\rangle$; rather than any explicit $Y_l^m(\theta, \phi)$ or Pauli spinors.

2. **Spherical Tensor Operators**

   (a) Prove the tensor operator composition laws

$$U^{k_1}_{q_1} V^{k_2}_{q_2} = \sum_{q,k} \langle k_1 q_1 k_2 q_2 \mid kq\rangle T^k_q$$

   and

$$T^k_q = \sum_{q_1,q_2} \langle kq \mid k_1 q_1 k_2 q_2\rangle U^{k_1}_{q_1} V^{k_2}_{q_2}$$

   (b) Show that $T^k_q \mid 0\rangle$, where $\mid 0\rangle$ is any $j = 0$ state, transforms under rotations like the angular momentum eigenstate $\mid kq\rangle$.

3. **Sakurai, p. 282, problem 5**
   Because of weak (neutral-current) interactions there is a parity-violating potential between the atomic electron and the nucleus as follows:

$$V = \lambda[\delta^{(3)}(\mathbf{x})\mathbf{S} \cdot \mathbf{p} + \mathbf{S} \cdot \mathbf{p}\delta^{(3)}(\mathbf{x})],$$

   where $\mathbf{S}$ and $\mathbf{p}$ are the spin and momentum operators of the electron, and the nucleus is assumed to be situated at the origin. As a result, the ground state of an alkali atom, usually characterized by $\mid n, l, j, m\rangle$ actually contains very tiny contributions from other eigenstates as follows:

$$\mid n, l, j, m\rangle \to \mid n, l, j, m\rangle + \sum_{n',l',j',m'} C_{n',l',j',m'} \mid n', l', j', m'\rangle.$$

On the basis of symmetry considerations *alone*, what can you say about (n', l', j', m'), which give rise to nonvanishing contributions? Suppose the radial wave functions and the energy levels are all known. Indicate how you may calculate $C_{n',l',j',m'}$. Do we get further restrictions on (n', l', j', m')?

4. **Angular momentum addition**
   We add two equal angular momenta $j_1 = j_2 = j$. Without using the symmetry properties of the Clebsch Gordan coefficients show that in any permutation of m_1 and m_2 the eigenfunctions of the total angular momentum are either symmetrical (invariant) or antisymmetrical (multiplied by -1), and that this symmetry property depends only on J (the total angular momentum). Show that they are symmetrical or antisymmetrical according as $(-1)^{2j+J}$ is equal to $+1$ or -1.

5. **Electric dipole moment**
   A charged particle with spin operator $\mathbf{S}$ is assumed to possess an electric dipole moment operator $\mu\mathbf{S}$, where μ is a numerical constant, so that the hamiltonian for this particle in any electric field $\mathbf{E}$ contains the interaction term $-\mu\mathbf{S}\cdot\mathbf{E}$. Show that neither space inversion or time reversal is a symmetry operation for this particle moving in a spherically symmetric electrostatic potential $\phi(r)$, even when no external electric field is present.

6. **Sakurai, problem 7, p. 283**

   (a) Let $\psi(\mathbf{x}, t)$ be the wave function of a spinless particle corresponding to a plane wave in three dimensions. Show that $\psi^*(\mathbf{x}, -t)$ is the wave function for the plane wave with the momentum direction reversed.

   (b) Let $\chi(\hat{\mathbf{n}})$ be the two-component eigenspinor of $\sigma\cdot\hat{\mathbf{n}}$ with eigenvalue $+1$. *Using the explicit form of* $\chi(\hat{\mathbf{n}})$ (in terms of the polar and azimuthal angles β and γ that characterize $\hat{\mathbf{n}}$) verify that $-i\sigma_2\chi^*(\hat{\mathbf{n}})$ is the two-component eigenspinor with the spin direction reversed.

7. **Sakurai, p. 283, problem 10**

   (a) What is the time-reversed state correspoonding to $\mathcal{D}(R)|\, j, m\rangle$?

   (b) Using the properties of time reversal and rotations, prove

   $$\mathcal{D}^{(j)}_{m'm}{}^*(R) = (-1)^{m-m'}\mathcal{D}^{(j)}_{-m',-m}(R).$$

   (c) Prove $\Theta|\, j, m\rangle = i^{2m}|\, j, -m\rangle$.