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Lecture 10

12 Neutron-Proton scattering

Consider low energy neutron proton scattering. The deuteron, the bound state of neutron and proton tellsus something about the interaction. The deuteron is the only bound state of neutron and proton. It has $J = 1$, binding energy of 2.23MeV. It has a quadrupole moment. As there is no $J = 0$ bound state, the interaction depends on spin. The most general state with spin 1 is

$$\Psi = a|\,^3S_1\rangle + b|\,^3P_1\rangle + c|\,^1P_1\rangle + d|\,^3D_1\rangle.$$

The magnetic moment of the deuteron is roughtly $\mu_n + \mu_p$. The contribution from $l = 1$ would be much larger so there can not be much $l = 1$ in the wave function. If the interaction is parity invariant then either $b = c = 0$ or $a = d = 0$. Must be that $b = c = 0$ in view of the small magnetic moment and $|D| \ll |a|$. Since there is a quadrupole moment, and we know that $l = 0$ is speherically symmetric, it must be that $|d| \neq 0$. The expectation value of the quadrupole meomtn will be

$$\langle Q\rangle = d\left\langle ^3S_1 \mid Q \mid ^3D_1\right\rangle + c.c. + O(d^2).$$

As there are both $l = 0$ and $l = 2$ contributions to the wave functions, the interatino does not conserve orbital angular moemntum, only total.

Write an interaction potential that distinguishes $l = 0$.

$$V = V_s(r)P_s + V_t(r)P_t,$$

where P_s and P_s are the singlet and triplet projection operators.

$$P_s = \frac{1}{4}(1 - \sigma_{\mathbf{n}} \cdot \sigma_{\mathbf{p}}), \qquad P_t = \frac{1}{4}(3 + \sigma_{\mathbf{n}} \cdot \sigma_{\mathbf{p}}) \tag{1}$$

Then

$$M(\mathbf{k}_f, \mathbf{k}_i) = f_s(k,\theta)P_s + f_t(k,\theta)P_t \tag{2}$$

For the unpolarizedbeam, the initial spin density matrix is $\phi_i = \frac{1}{4}$. The differentiual cross section is

$$\frac{d\sigma}{d\Omega} = \frac{1}{4}\mathrm{Tr}(P_s|f_s|^2 + P_t|f_t|^2) \tag{3}$$

$$= \frac{1}{4}(|f_s|^2 + 3P_t|f_t|^2) \tag{4}$$

After scattering the most general spin-density matrix is

$$\rho = \frac{1}{4}\left(1 + \sigma_{\mathbf{n}} \cdot \mathbf{P_n} + \sigma_{\mathbf{p}} \cdot \mathbf{P_p} + \sum_{i,j=1}^{3} \sigma_{n,i}\sigma_{p,j}C_{ij}\right) \tag{5}$$

We find if the initial spin state is random

$$\rho_f = \frac{1}{4}\frac{|f_s|^2 P_s + |f_t|^2 P_t}{d\sigma/d\Omega} = \frac{d\sigma/d\Omega + \frac{1}{4}\sigma_{\mathbf{n}} \cdot \sigma_{\mathbf{p}}(|f_t|^2 - |f_s|^2)}{4(d\sigma/d\Omega)} \tag{6}$$

and

$$C_{ij} = \delta_{ij}\frac{|f_t|^2 - |f_s|^2}{4(d\sigma/d\Omega)}$$

13 Low energy scattering

Consider s-wave scattering from a square well potential with $V = V_0$ for $r < R$ and zero otherwise. The general solution for a free particle (outside the potential) for each partial wave

$$A_l(r) = e^{i\delta_l}[\cos\delta_l j_l(kr) - \sin\delta_l n_l(kr)] \tag{7}$$

and for s-wave

$$A_0(r) = e^{i\delta_0}[\cos\delta_0 j_0(kr) - \sin\delta_0 n_0(kr)] = e^{i\delta_0}\sin(kr + \delta_0) \tag{8}$$

At $r < R$, with the requirement that $u(0) = 0$, solutions to the Schrodinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + V_0 = E \tag{9}$$

are $u(r) = A\sin k'r$ with $k' = \sqrt{2m(E - V_0)}/\hbar$. Then match logarithmic derivative at the boundary $r = R$.

$$\begin{aligned}
k'\cot k'R &= k\cot(kR + \delta_0) && (10)\\
\rightarrow k'\tan(kR + \delta_0) &= k\tan k'R && (11)\\
\rightarrow \delta_0 &= \tan^{-1}(\frac{k}{k'}\tan(k'R)) - kR && (12)
\end{aligned}$$

Since the scattering amplitude for $l = 0$ is

$$f(\theta) = \frac{1}{2ik}e^{i\delta_0}\sin\delta_0 \tag{13}$$

the differential cross section becomes

$$\frac{d\sigma}{d\Omega} = \frac{1}{4k^2}\sin^2\delta_0$$

If the potential is attractive, $V_0 = -|V_0|$, the cross section will reach a maximum whenever $\delta_0 = (n + \frac{1}{2})\pi$. If the potential is attractive, $V_0 < 0$, and $k' < k$, then as we increase the energy from zero, the cross section will go through resonances as δ_0 goes through $(n + \frac{1}{2})\pi$.

For very low energy, $k \rightarrow 0$, from Equation 12 we get

$$\begin{aligned}
\delta_0 + kR &\sim \frac{k}{k'}\tan(k'R)\\
k(-a + R) &\sim \frac{k}{k'}\tan(k'R)\\
\rightarrow a &\sim R - \frac{\tan k'R}{k'}
\end{aligned}$$

Then the cross section is independent of energy and

$$\sigma = 4\pi\frac{1}{k^2}\sin\delta_0^2 = 4\pi a^2 \tag{14}$$

If $k'R \sim \pi/2$ then the cross section can be very large at very low energy.

13.1 More low energy scattering

Again we consider s-wave. Same as above but slightly different detail. This time we suppose that the energy is very slightly negative, so that the particle cannot quite propagate. Inside the well with potential $V(r) = -V_0$ for $r < R$ (and zero outside),

$$-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}u - V_0 u = Eu$$

$E < 0$, and $|E| \ll |V|_0$. E is the binding energy. Then

$$\frac{d^2u}{dr^2} = -\frac{2m}{\hbar^2}(V + E) = -K^2$$

$$\rightarrow R(r) = \frac{\sin Kr}{r}$$

Outside the well, $R(r) \sim \frac{e^{-\alpha r}}{r}$ where $\alpha = \sqrt{2mE}/\hbar$. At the boundary, $r = R$,

$$K \cot KR = -\alpha$$

The deuteron is weakly bound so $\alpha^2 \ll K$ and $K_0^2 = \frac{2m}{\hbar^2}V_0$. And as α is small, $KR \rightarrow \pi/2$ from above so that α is positive. In a tightly bound state, $KR \sim \pi$ so that $R(r)$ goes to nearly zero at the boundary. But this is a weakly bound state so KR is just a little bit bigger than $\pi/2$ so that it is just turning over at the boundary where it joins to the exponentially falling outside wave function.

The scattering state, where $E > 0$ and $|E| \ll |V_0|$ will have the inside solution

$$\frac{\sin(Kr)}{r}$$

and outside

$$\frac{\sin(kr + \delta)}{r}$$

The boundary condition gives

$$K \tan(kR + \delta) = k \tan(KR)$$

where $k^2 = 2m(V_0 + E)/\hbar^2$. Now KR is just a bit less than $\pi/2$ so that there is an oscillatory wave function outside the boundary.

If $\delta = -ka$ then in the limit where $k \rightarrow 0$,

$$\begin{aligned} k(R - a) &\sim \frac{k}{K}\tan KR \\ a &\sim R(1 - \frac{1}{RK \cot RK}) \end{aligned}$$

a is the scattering length. The cross section (we are assuming only s-wave scattering) is

$$\sigma = \frac{4\pi}{k^2}\sin^2\delta_0 = \frac{4\pi}{k^2}\sin^2(ka) \rightarrow 4\pi a^2$$

We see that there is a pole in the cross section as RK goes through $\pi/2$

Next let's see if we can relate the scattering length a to the bound state energy. The radial wave function outside is

$$rR(r) \sim \sin(k(r-a)) \rightarrow k(r-a)$$

in the limit of small k. The wave function for the weakly bound state is

$$rR(r) \sim e^{-\alpha r} \rightarrow 1 - \alpha r.$$

It looks the same as the scattered state wave function in the $k \rightarrow 0$ limit. Match the logarithmic derivative

$$-\alpha = \frac{1}{r-a}.$$

Where to evaluate? Let's take $r = 0$. Then $\alpha = 1/a = \sqrt{2mE}/\hbar$. Or even better, note that

$$(1-\alpha r) \rightarrow \frac{k}{\alpha}(1-\alpha r) = \frac{k}{\alpha} - rk = k(r-a) \rightarrow a = -\frac{1}{\alpha}.$$

Then the binding energy is

$$E = \frac{\hbar^2}{2ma^2}$$

By measuring the cross section at very low energy, we can determine the binding energy.

14 Eikonal - short wavelength - approximation

In the high energy (short wavelength) limit, where the wavelength is short compared to the distance over which the potential varies significantly, we approximate the solution to the Schrodinger Equation as

$$\psi \sim e^{iS(x)/\hbar} \tag{15}$$

where $S(x)$, the classical action, is defined by the Hamliton-Jacobi equation. (If we substitute Equation 15 into the Schrodinger equation and just keep lowest order in $\hbar$, we have the Hamilton-Jacobi equation for S.

$$\frac{(\nabla S)^2}{2m} + V = E = \frac{\hbar^2 k^2}{2m} \tag{16}$$

We in principle solve the classical equations of motion to get S, but that could be difficult, and instead we approximate that motion as a straight line in the $+z$ direction at impact parameter b. We integrate to get S along the classical trajectory,

$$S = \int \sqrt{2m(E-V)}ds \tag{17}$$

and approximate the trajectory as the straight line along z with impact parameter b, reasonable in the high energy limit. Then

$$S(z) = \hbar \int_{-\infty}^{z} \left[(k^2 - \frac{2m}{\hbar^2} V(\sqrt{b^2 + z'^2}) \right]^{1/2} dz' + \text{constant} \tag{18}$$

We chose the constant so that in the $V \to 0$ limit, $S(z) = \hbar kz$.

$$\begin{aligned} S(z) &= \hbar kz + \hbar \int_{-\infty}^{z} \left[\left[(k^2 - \frac{2m}{\hbar^2} V(\sqrt{b^2 + z'^2}) \right]^{1/2} - k \right] dz' \\ &= \hbar kz + \hbar \int_{-\infty}^{z} \left[k \left[(1 - \frac{2m}{k^2\hbar^2} V(\sqrt{b^2 + z'^2}) \right]^{1/2} - k \right] dz' \\ &\sim \hbar kz + \hbar \int_{-\infty}^{z} \left[k \left[(1 - \frac{m}{k^2\hbar^2} V(\sqrt{b^2 + z'^2}) \right] - k \right] dz' \\ &\sim \hbar kz - \hbar \int_{-\infty}^{z} \frac{m}{k\hbar^2} V(\sqrt{b^2 + z'^2}) dz' \end{aligned}$$

Finally

$$\psi = \frac{1}{(2\pi)^{3/2}} e^{ikz} \exp \int_{-\infty}^{z} -i \frac{m}{k\hbar^2} V(\sqrt{b^2 + z'^2}) dz'$$

Then we can write the scattering amplitude

$$
\begin{aligned}
f(\mathbf{k}',\mathbf{k}) &= -\frac{4\pi^2 m}{\hbar^2}\int d^3\mathbf{x}'\frac{e^{-i\mathbf{k}'\cdot\mathbf{x}'}}{(2\pi)^{3/2}}V(\mathbf{x}')\psi \\
&= -\frac{4\pi^2 m}{\hbar^2}\int d^3\mathbf{x}'\frac{e^{-i\mathbf{k}'\cdot\mathbf{x}'}}{(2\pi)^{3/2}}V(\mathbf{x}')\frac{e^{i\mathbf{k}\cdot\mathbf{x}'}}{(2\pi)^{3/2}}\exp\int_{-\infty}^{z} -\frac{im}{k\hbar^2}V(\sqrt{b^2+z'^2})dz' \\
&= -\frac{m}{2\pi\hbar^2}\int d^3x' e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}'}V(\sqrt{b^2+z'^2})\exp\int_{-\infty}^{z} -\frac{im}{k\hbar^2}V(\sqrt{b^2+z''^2})dz''
\end{aligned}
$$

Consider $(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}' = (k\hat{\mathbf{z}}-\mathbf{k}')\cdot(\mathbf{b}+z\hat{\mathbf{z}}) \sim -\mathbf{k}'\cdot\mathbf{b}$, assuming the scattering angle is small ($\mathbf{k}$ is nearly parallel to $\mathbf{k}'$.) The component of $\mathbf{k}'$ in the radial direction is $k\sin\theta$. The component of $\mathbf{b}$ in the scattering plane is $\cos\phi$. So $\mathbf{k}'\cdot\mathbf{b} = k\sin\theta\cos\phi \sim k\theta\cos\phi$ since the scattering angle is small.

$$
\begin{aligned}
f(\mathbf{k}',\mathbf{k}) &= -\frac{m}{2\pi\hbar^2}\int dz' bdbd\phi e^{ikb\theta\cos\phi}V(\sqrt{b^2+z'^2})\exp\int_{-\infty}^{z'} -\frac{im}{k\hbar^2}V(\sqrt{b^2+z''^2})dz'' \\
&= -\frac{m}{2\pi\hbar^2}\int bdbd\phi e^{ikb\theta\cos\phi}\int dz' V(\sqrt{b^2+z'^2})\exp\int_{-\infty}^{z'} -\frac{im}{k\hbar^2}V(\sqrt{b^2+z''^2})dz'' \\
&= \frac{k\hbar^2}{im}\frac{m}{2\pi\hbar^2}\int bdbd\phi e^{ikb\theta\cos\phi}\int dz'\frac{\partial}{\partial z'}\exp\int_{-\infty}^{z'} -\frac{im}{k\hbar^2}V(\sqrt{b^2+z''^2})dz'' \\
&= \frac{k\hbar^2}{im}\frac{m}{2\pi\hbar^2}\int bdbd\phi e^{ikb\theta\cos\phi}\exp\int_{-\infty}^{z'} -\frac{im}{k\hbar^2}V(\sqrt{b^2+z''^2})dz''|_{-\infty}^{\infty} \\
&= -i\frac{k}{2\pi}\int bdbd\phi e^{ikb\theta\cos\phi}\exp\int_{-\infty}^{z'} -\frac{im}{k\hbar^2}V(\sqrt{b^2+z''^2})dz''|_{-\infty}^{\infty}
\end{aligned}
$$

The Bessel density

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi} e^{ix\cos\tau}d\tau$$

allows us to rewrite

$$f(\mathbf{k}',\mathbf{k}) = -ik\int bdbJ_0(kb\theta)e^{-i\frac{m}{k\hbar^2}\int_{-\infty}^{z}V(z')dz'|_{-\infty}^{\infty}}$$

Note that

$$\Delta(b) = \frac{m}{2k\hbar^2}\int_{-\infty}^{z}V(z')dz'|_{-\infty}^{\infty} = \frac{m}{2k\hbar^2}\int_{-\infty}^{\infty}V(z')dz' - 0. \tag{19}$$

Then

$$f(\mathbf{k}',\mathbf{k}) = -ik\int bdbJ_0(kb\theta)(e^{2i\Delta(b)}-1)$$

which looks a lot like the partial wave amplitude **??**

$$f(\theta) \;=\; \frac{1}{k}\sum_l (2l+1)e^{i\delta_l}\sin\delta_l P_l(\cos\theta) = \frac{1}{2ik}\sum_l (2l+1)(e^{2i\delta_l}-1)P_l(\cos\theta)$$

14.1 Eikonal and partial waves

In the high energy regime many partial waves contribute up to $l = kb$ where b is the impact parameter. In the sum

$$f(\theta) = \frac{1}{k}\sum_l (2l+1)e^{i\delta_l}\sin\delta_l P_l(\cos\theta),$$

we can make the substitution

$$\sum_l^{l_{max}} \to k\int db, \qquad P_l(\cos\theta) \to J_0(l\theta) = J_0(kb\theta)$$

that latter in the limit where l is large and θ is small and $\delta_l \to \Delta(b)$ where $b = l/k$. This will be an OK approximation for large l, that is $l > Rk$ where R is the range of the potential. Then

$$e^{2i\delta_l} - 1 = e^{2i\Delta(b)} - 1 = 0$$

for $l > l_{max}$

$$\begin{aligned} f(\theta) &= \frac{1}{2ik}\int kdb2b(e^{2i\Delta(b)}-1)J_0(kb\theta) \\ &= -i\int kbdbJ_0(kb\theta)(e^{-2i\Delta(b)}-1) \end{aligned}$$