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Lecture X

1.2 Partial Wave Analysis

We have described scattering in terms of an incoming plane wave, a momentum eigenket, and and outgoing spherical wave, also with definite momentum. We now consider the basis of free particle states with definite energy and angular momentum (rather than linear momentum) that look like $\mid E, l, m\rangle$. These are eigenkets of of H_0, L^2, and L_z. We would like to expand our plane wave in terms of these spherical waves like so

$$\mid \mathbf{k}\rangle = \sum_{l,m} \mid E, l, m\rangle\langle E, l, m \mid \mathbf{k}\rangle \tag{1.1}$$

Then we can write the scattering amplitude

$$\begin{aligned} f(\mathbf{k}', \mathbf{k}) &= -\frac{1}{4\pi}\frac{2m}{\hbar^2}8\pi^3 \langle \mathbf{k}' \mid T \mid \mathbf{k}\rangle \\ &= -\frac{1}{4\pi}\frac{2m}{\hbar^2}8\pi^3 \int dE' \sum_{l',m'} \int dE \sum_{l,m} \langle \mathbf{k}' \mid E, l', m'\rangle \langle E, l', m' \mid T \mid E, l, m\rangle \langle E, l, m \mid \mathbf{k}\rangle \end{aligned}$$

If the scattering potential is spherically symmetric, T is a scalar operator, and by WE, $l = l', m = m'$, and $\langle E, l, m \mid T \mid E, l, m\rangle$ is independent of m. Then

$$\begin{aligned} f(\mathbf{k}', \mathbf{k}) &= -\frac{1}{4\pi}\frac{2m}{\hbar^2}8\pi^3 \int\int dE dE' \sum_{l,m} \langle \mathbf{k}' \mid E', l, m\rangle \langle E', l, m \mid T \mid E, l, m\rangle \langle E, l, m \mid \mathbf{k}\rangle \\ &= -\frac{1}{4\pi}\frac{2m}{\hbar^2}8\pi^3 \int\int dE dE' \sum_{l,m} \langle \mathbf{k}' \mid E', l, m\rangle T_l \langle E, l, m \mid \mathbf{k}\rangle \end{aligned} \tag{1.2}$$

$$\tag{1.3}$$

The "spherical" scattering amplitude conserves angular momentum.

Now let's figure out $\langle \mathbf{k} \mid E, l, m\rangle$. Consider the state $\mid k\hat{\mathbf{z}}\rangle$.

$$\begin{aligned} \langle k\hat{\mathbf{z}} \mid L_z \mid E, l, m\rangle &= 0 \quad (m \neq 0) \\ \rightarrow \langle k\hat{\mathbf{z}} \mid E, l, m\rangle &= 0 \quad (m \neq 0) \end{aligned}$$

Also $\langle k\hat{\mathbf{z}} \mid E, l, m = 0\rangle$ is independent of θ, ϕ,so $\langle k, \hat{\mathbf{z}} \mid E, l, m = 0\rangle = \sqrt{\frac{2l+1}{4\pi}} g_l(k)$. We can transform the z-direction momentum ket into an arbitrary direction by a rotation.

$$\mid \mathbf{k}\rangle = \mathcal{D}(\alpha = \phi, \beta = \theta, 0)\mid k\hat{\mathbf{z}}\rangle \tag{1.4}$$

Then

$$\begin{aligned} \langle \mathbf{k} \mid E, l, m\rangle &= \langle k\hat{\mathbf{z}} \mid \mathcal{D} \mid E, l, m\rangle \\ &= \sum_{l'} \langle k\hat{\mathbf{z}} \mid E, l', m' = 0\rangle \langle E, l', m' = 0 \mid \mathcal{D} \mid E, l, m\rangle \end{aligned}$$

$$
\begin{aligned}
&= \sum_{l'} \sqrt{\frac{2l+1}{4\pi}} g_l(k) \mathcal{D}^l_{0,m} \\
&= \sum_{l'} \sqrt{\frac{2l+1}{4\pi}} g_l(k) \sqrt{\frac{4\pi}{2l+1}} Y_l^m(\theta,\phi) \\
&= \sum_{l} g_l(k) Y_l^m(\theta,\phi)
\end{aligned}
$$

One more thing.

$$
\begin{aligned}
\langle \mathbf{k} \mid H_0 - E \mid E, l, m\rangle &= \langle \mathbf{k} \mid E, l, m\rangle (\frac{\hbar^2 k^2}{2m} - E) = 0 \\
\rightarrow \langle \mathbf{k} \mid E, l, m\rangle &\propto \delta(\frac{\hbar^2 k^2}{2m} - E) \\
\rightarrow g_l(k) &= N_l \delta(\frac{\hbar^2 k^2}{2m} - E)
\end{aligned}
$$

To determine N_l let's try to normalize.

$$
\begin{aligned}
\langle E', l', m' \mid E, l, m\rangle &= \int d^3k \langle E', l', m' \mid \mathbf{k}\rangle\langle \mathbf{k} \mid E, l, m\rangle \\
&= \int k''^2 dk'' d\Omega N_{l'}^* N_l Y_{l'}^{m'*} Y_l^m \delta(\frac{\hbar^2 k''^2}{2m} - E')\delta(\frac{\hbar^2 k''^2}{2m} - E) \\
&= \int k''^2 dk'' |N_l|^2 \delta(\frac{\hbar^2 k''^2}{2m} - E')\delta(\frac{\hbar^2 k''^2}{2m} - E) \\
&= \int \frac{k''m}{\hbar^2} dE'' |N_l|^2 \delta(\frac{\hbar^2 k''^2}{2m} - E')\delta(\frac{\hbar^2 k''^2}{2m} - E) \\
&= \frac{k''m}{\hbar^2} |N_l|^2 \delta(E - E')\delta_{ll'}\delta_{mm'} \\
\rightarrow g_l &= \frac{\hbar}{\sqrt{k''m}} \delta(E - \frac{\hbar^2 k''^2}{2m})
\end{aligned}
$$

From which we get

$$
\langle \mathbf{k} \mid E, lm,\rangle = \frac{\hbar}{\sqrt{km}} Y_l^m(\theta,\phi)\delta(E - \frac{\hbar^2 k^2}{2m}) \tag{1.5}
$$

1.2.1 $\rho_0 \rightarrow \pi\pi$

The ρ meson is spin 1 and it decays to two spin 0 pions. Suppose that the ρ is in the $l = 1, m = 1$ state, where there is some z-axis defined by something. The final state has the same angular momentum quantum numbers and the amplitude to find a π with momentum in the $\hat{\mathbf{k}}$ direction is

$$
\langle \mathbf{k} \mid E, l, m\rangle \propto Y_1^1(\hat{\mathbf{k}}) \propto \sin\theta
$$

The angular distribution of the π is

$$
|Y_1^1|^2 \sim \sin^2\theta
$$

If we imagine producing ρ in e^+e^- collisions where electrons and positrons are polarized so that $j_z = +1$ along the z-axis defined by the direction of the positron beam, then

$$\frac{d\sigma}{d\Omega}(\theta) \propto \sin^2\theta$$

1.2.2 Back to partial wave expansion

Substituting into Equation 1.29 we have

$$\begin{aligned}
f(\mathbf{k}',\mathbf{k}) &= -\frac{1}{4\pi}\frac{2m}{\hbar^2}8\pi^3\int dE\int dE'\sum_{l,m}\sum_{l',m'}\langle\mathbf{k}'\mid E',l',m'\rangle\langle E',l',m'\,|T_l|\,E,l,m\rangle\langle E,l,m\mid\mathbf{k}\rangle\\
&= -\frac{1}{4\pi}\frac{2m}{\hbar^2}8\pi^3\int dE\int dE'\sum_{l,m}\langle\mathbf{k}'\mid E',l,m\rangle T_l\langle E,l,m\mid\mathbf{k}\rangle\\
&= -\frac{1}{4\pi}\frac{2m}{\hbar^2}8\pi^3\int dE\int dE'\sum_{l,m}\frac{\hbar}{\sqrt{k'm}}Y_l^m(\mathbf{k}')\delta(E'-\frac{\hbar^2k'^2}{2m})T_l\frac{\hbar}{\sqrt{km}}Y_l^{m*}(\mathbf{k})\delta(E-\frac{\hbar^2k^2}{2m})\\
&= -\frac{1}{4\pi}\frac{2m}{\hbar^2}8\pi^3\sum_{l,m}\frac{\hbar^2}{km}Y_l^m(\mathbf{k}')Y_l^m(\mathbf{k})^*T_l\\
&= -\frac{4\pi^2}{k}\sum_{l,m}Y_l^m(\mathbf{k}')Y_l^m(\mathbf{k})^*T_l
\end{aligned}$$

Let $\mathbf{k} = |k|\hat{\mathbf{z}}$ so that $\theta = 0, \phi = ?$ and then $Y_l^m(\mathbf{k}) = \sqrt{\frac{2l+1}{4\pi}}\delta_{m0}$. So only $m = 0$ contributes. Then $Y_l^0(\mathbf{k}') = \sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta)$ where θ is the angle between $\mathbf{k}$ and $\mathbf{k}'$. The scattering amplitude becomes

$$f(\mathbf{k}',\mathbf{k}) = -\frac{4\pi^2}{k}\sum_l\frac{2l+1}{4\pi}P_l(\theta)T_l = -\frac{\pi}{k}\sum_l(2l+1)P_l(\cos\theta)T_l \tag{1.6}$$

Define $f_l(k) = -\frac{\pi T_l(E)}{k}$ and

$$f(\mathbf{k}',\mathbf{k}) = \sum_l(2l+1)P_l(\cos\theta)f_l(k) \tag{1.7}$$

$f_l(k)$ is amplitude to scatter an incident particle with angular momentum $\hbar l$ or impact parameter b such that $kb = l$. Remember that the outgoing solution to the SE far outside the range of the potential is

$$\psi^+ = \frac{1}{(2\pi)^{3/2}})\left[e^{ikz} + f(\theta)\frac{e^{ikr}}{r}\right]$$

1.2.3 Expansion of plane wave as spherical waves

The radial part of the free particle Schrodinger equation is

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2}+\left[V(r)+\frac{l(l+1)\hbar^2}{2mr^2}\right]u=Eu$$

The solution to the free particle Schrodinger equation in spherical coordinates is

$$\langle \mathbf{x} \mid E,l,m\rangle = c_l j_l(kr)Y_l^m(\hat{\mathbf{r}}).$$

Next expand the plane wave as a linear combination of incoming and outgoing spherical waves.

$$\begin{aligned}\langle \mathbf{x} \mid \mathbf{k}\rangle &= \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}} = \sum_{l,m}\int dE\langle \mathbf{x} \mid E,l,m\rangle\langle E,l,m \mid \mathbf{k}\rangle \\ &= \sum_{l,m}\int dE c_l j_l(kr)Y_l^m(\hat{\mathbf{r}})\frac{\hbar}{\sqrt{mk}}\delta(E-\frac{\hbar^2k^2}{2m})Y_l^m(\hat{\mathbf{k}}) \\ &= \sum_l \frac{(2l+1)}{4\pi}P_k(\hat{\mathbf{k}}\cdot\hat{\mathbf{r}})\frac{\hbar}{\sqrt{mk}}c_l j_l(kr)\end{aligned}$$

where we use the addition theorem

$$\sum_m Y_l^m(\hat{\mathbf{r}})Y_l^{m*}\hat{\mathbf{k}}) = \frac{2l+1}{4\pi}P_l(\hat{\mathbf{k}}\cdot\hat{\mathbf{r}})$$

Turns out that $c_l = \frac{i^l}{\hbar}\sqrt{\frac{2mk}{\pi}}$ so that

$$e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_l (2l+1)i^l j_l(kr)P_l(\hat{\mathbf{k}}\cdot\hat{\mathbf{r}})$$

As $r\to\infty$,

$$e^{i\mathbf{k}\cdot\mathbf{x}} \to \sum_l (2l+1)\frac{e^{i(kr))}-e^{-i(kr-(l\pi))}}{2ikr}P_l(\cos\theta) \tag{1.8}$$

1.2.4 Partial wave expansion

Now

$$\psi^+ = \frac{1}{(2\pi)^{3/2}})\left[e^{ikz}+f(\theta)\frac{e^{ikr}}{r}\right] \tag{1.9}$$

$$\psi^+ = \frac{1}{\sqrt{8\pi^3}}\sum_l i^l(2l+1)A_l(r)P_l(\cos\theta) \tag{1.10}$$

$$= \frac{1}{\sqrt{8\pi^3}}\sum_l i^l(2l+1)(c_l^1h_l^1(r)+c_l^2h_l^2(r))P_l(\cos\theta) \tag{1.11}$$

Remember that for large r,

$$h_l^1 \to \frac{e^{i(kr-(l\pi/2)}}{ikr}, \qquad h_l^2 \to \frac{e^{-i(kr-(l\pi/2)}}{ikr} \tag{1.12}$$

becomes, using

$$j_l(kr) \to (large\ r) \to \frac{e^{i(kr-(l\pi/2))} - e^{-i(kr-(l\pi/2))}}{2ikr}$$

Anyway, we can write, the general solution to the Schrodinger equation in the partial wave basis, far from the scattering potential as

$$\begin{aligned} \psi_{elastic} &= \psi^+ = \frac{1}{\sqrt{8\pi^3}} \sum_l i^l (2l+1) \frac{\eta_l e^{i(kr-(l\pi/2))} - c_l e^{-i(kr-(l\pi/2))}}{2ikr} \\ &= \psi^+ = \frac{1}{\sqrt{8\pi^3}} \sum_l (2l+1) \frac{\eta_l e^{i(kr))} - c_l e^{-i(kr-(l\pi))}}{2ikr} \end{aligned}$$

Then since

$$\psi_{elastic} - \frac{1}{(2\pi)^{3/2}} e^{i\mathbf{x}\cdot\mathbf{k}} = \frac{1}{(2\pi)^{3/2}} f(\theta) \frac{e^{ikr}}{r}$$

we know that $c_l = 1$ so that the ingoing wave is the same. Therefore

$$\psi_{elastic} = \psi^+ = \frac{1}{\sqrt{8\pi^3}} \frac{1}{2ikr} \sum_l i^l (2l+1)(\eta_l e^{i(kr-l\pi/2)} - e^{-i(kr-l\pi/2)})$$

Probablity conservation requres $|\eta_l| < 1$. If $|\eta_l| = 1$, the scattering is pure elastic and each partial wave gets some phase shift. If $\eta_l = 0$ the scattering for that partial wave is purely inelastic.

And we know that

$$\frac{\eta_l - 1}{2ik} = f_l$$

$$\begin{aligned} \psi^+ &= \frac{1}{(2\pi)^{3/2})} \left[\sum_l (2l+1) P_l(\cos\theta) \left(\frac{e^{ikr} - e^{-i(kr-l\pi)}}{2ikr} \right) + f(\theta) \frac{e^{ikr}}{r} \right] \\ &= \frac{1}{(2\pi)^{3/2})} \left[\sum_l (2l+1) P_l(\cos\theta) \left(\frac{e^{ikr} - e^{-i(kr-l\pi)}}{2ikr} \right) + \sum_l (2l+1) P_l(\cos\theta) f_l(k) \frac{e^{ikr}}{r} \right] \\ &= \frac{1}{(2\pi)^{3/2})} \left[\sum_l (2l+1) P_l(\cos\theta)(1 + 2ikf_l(\theta)) \left(\frac{e^{ikr}}{2ikr} \right) + \sum_l (2l+1) P_l(\cos\theta) \frac{e^{-i(kr-l\pi)}}{2ikr} \right] \end{aligned}$$

Unitarity requires that flux is conserved for each angular momentum state. Outgoing flux is no more than incoming. Therefore

$$|1 + 2ikf_l(\theta)| = |\eta_l| \leq 1 \tag{1.13}$$

and equal to one for elastic scattering.

For elastic scattering we define a phase shift

$$1 + 2ikf_l(\theta) = e^{2i\delta_l} \tag{1.14}$$

The elastic partial wave amplitude

$$f_l(\theta) = \frac{e^{2i\delta_l} - 1}{2ik} = \frac{e^{i\delta_l}\sin\delta_l}{k}$$

and

$$f(\theta) = \frac{1}{k}\sum_l (2l+1)e^{i\delta_l}\sin\delta_l P_l(\cos\theta) \tag{1.15}$$

The total cross section is

$$\sigma_t = \int d\Omega \sum_l (2l+1)^2 |f_l|^2 P_l^2(\cos\theta) \tag{1.16}$$

$$= 4\pi \sum_l (2l+1)|f_l|^2 \tag{1.17}$$

$$= \frac{\pi}{k^2}\sum_l (2l+1)|\eta_l - 1|^2 = \frac{\pi}{k^2}\sum_l (2l+1)(|\eta_l|^2 + 1 - 2\Re(\eta_l)) \tag{1.18}$$

(1.19)

If the scattering is elastic then the total cross section is

$$\sigma_{ela} = \frac{1}{k^2}\int d\Omega \sum_l (2l+1)^2 \sin^2\delta_l P_l^2(\cos\theta)$$
$$= \frac{4\pi}{k^2}\sum_l (2l+1)\sin^2\delta_l$$

where $\eta_l = e^{2i\delta_l}$. Suppose that there is an inelastic component, so that the magnitude of the outgoing wave at momentum k in ψ_{scat} is less than the magnitude in the incoming plane wave. Then $|\eta_l| < 1$. The inelastic cross section is the piece lost from the outgoing, namely $1 - |\eta_l|^2$. Therefore

$$\sigma_{inelastic} = 4\pi\sum_l (2l+1)\frac{(1-|\eta_l|^2)}{|2ik|^2} = \frac{\pi}{k^2}(2l+1)(1-|\eta_l|^2) \tag{1.20}$$

And the optical theorem ?

$$f(\theta) = \sum_l (2l+1)\frac{\eta_l - 1}{2ik}P_l(\cos\theta)$$
$$\mathrm{Im} f(0) = -\sum_l (2l+1)\Re\left(\frac{\eta_l - 1}{2k}\right)$$
$$\sigma_{tot} = -\frac{2\pi}{k^2}\sum_l (2l+1)\Re(\eta_l - 1)$$

The inelastic cross section is the difference of the total and the elastic

$$\sigma_{ine} = \sigma_{tot} - \sigma_{elas}$$
$$= -\frac{2\pi}{k^2}\sum_l (2l+1)\Re(\eta_l - 1) - \frac{\pi}{k^2}\sum_l (2l+1)(|\eta_l|^2 + 1 - 2\Re(\eta_l))$$
$$= \frac{\pi}{k^2}\sum_l (2l+1)(1-|\eta_l|^2)$$

1.2.5 General solution

Far from the scattering center where $V \to 0$, the solution to the free particle Schrodinger equation is a linear combination of spherical bessel functions.

$$\psi^+ = \frac{1}{(2\pi)^{3/2}} \sum_l i^l (2l+1) A_l(r) P_l(\cos\theta) \tag{1.21}$$

$$= \frac{1}{(2\pi)^{3/2}} \sum_l i^l (2l+1)(c_l^1 h_l^1(r) + c_l^2 h_l^2(r)) P_l(\cos\theta) \tag{1.22}$$

Remember that for large r,

$$h_l^1 \to \frac{e^{i(kr-(l\pi/2)}}{ikr}, \qquad h_l^2 \to \frac{e^{-i(kr-(l\pi/2)}}{ikr} \tag{1.23}$$

Meanwhile, in terms of the scattering amplitude and phase shift we write

$$\psi^+ = \frac{1}{(2\pi)^{3/2}} \sum_l (2l+1) P_l \left[\frac{e^{2i\delta_l} e^{ikr}}{2ikr} - \frac{e^{-i(kr-l\pi)}}{2ikr} \right] \tag{1.24}$$

Comparison with Equation 1.49 gives $c_l^1 = \frac{1}{2} e^{2i\delta_l}$ and $c_l^2 = \frac{1}{2}$. Then

$$\begin{aligned} A_l(r) &= c_l^1 h_l^1(r) + c_l^2 h_l^2(r) = \frac{1}{2} e^{2i\delta_l} (j_l + i n_l) + \frac{1}{2} (j_l - i n_l) \\ &= \frac{1}{2} e^{i\delta_l} \left((\cos\delta_l + i \sin\delta_l)(j_l + i n_l) + (\cos\delta_l - i\sin\delta_l)(j_l - i n_l) \right) \\ &= e^{i\delta_l} (\cos\delta_l j_l - \sin\delta_l n_l) \end{aligned}$$

1.2.6 Hard Sphere scattering

The probability of finding the particle inside the radius R of the sphere is zero. It is zero at the boundary. So

$$\psi^+(R) = j_l(kR)\cos\delta_l - n_l(kR)\sin\delta_l = 0 \tag{1.25}$$

$$j_l(kR)\cos\delta_l = n_l(kR)\sin\delta_l \rightarrow \tan\delta_l = \frac{j_l(kR)}{n_l(kR)} \tag{1.26}$$

As we have determined all of the phase shifts, we can compute the exact differential cross section. Let's consider the limits. Suppose that the energy of the incoming particle is very low. So low that the angular momentum $\hbar kR \ll \hbar$. Then only the $l = 0$ partial wave will contribute. Partial waves with $l > 0$ have zero amplitude of appearing at R. Also consider the small x limit of $j_l(x), n_l(x)$.

$$\begin{aligned}
\lim_{x\rightarrow 0} j_l(x) &= \frac{2^l l!}{(2l+1)!}x^l \\
\lim_{x\rightarrow 0} n_l(x) &= -\frac{(2l)!}{2^l l!}\frac{1}{x^{l+1}} \\
\rightarrow \lim_{kR\rightarrow 0} \tan\delta_l &\sim \frac{(2^l l!)^2}{(2l+1)(2l)!}x^{(2l+1)}
\end{aligned}$$

Clearly the $l = 0$ partial wave is the most important component of the wave function when $kR \ll 1$.

The $l = 0$ phase shift is

$$\tan\delta_0 = \frac{\sin kR/kR}{\cos kR/KR} = -\tan kR \rightarrow \delta_0 = kR \tag{1.27}$$

The differential cross section for s-wave scattering is

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = |\frac{1}{k}\sin kR|^2 \sim R^2 \tag{1.28}$$

No angular dependence of course. The total cross section is $\sigma_t = 4\pi R^2$, four times the geometric cross section.

In the limit of high energy, where $kR \gg 1$, we can use the asymptotic form for j_l and n_l. Then all partial waves with $l \leq kR$ will contribute.

$$\lim_{kR\rightarrow\infty} j_l(kR) = \frac{\sin(kR - l\frac{\pi}{2})}{kR} \tag{1.29}$$

$$\lim_{kR\rightarrow\infty} n_l(kR) = -\frac{\cos(kR - l\frac{\pi}{2})}{kR} \tag{1.30}$$

$$\tag{1.31}$$

Then

$$\lim_{kR\rightarrow\infty} \tan\delta_l = -\tan(kR - l\frac{\pi}{2}) \tag{1.32}$$

$$\sin^2\delta_l = \sin^2(kR - l\frac{\pi}{2}) \tag{1.33}$$

$$\int d\Omega|f(\theta)|^2 = \frac{4\pi}{k^2}\sum_l (2l+1)\sin^2(kR - l\frac{\pi}{2}) \tag{1.34}$$

So let's sum to kR.

$$\int d\Omega |f(\theta)|^2 = \frac{4\pi}{k^2}\sum_{l=0}^{kR}(2l+1)\sin^2(kR - l\frac{\pi}{2}) \tag{1.35}$$
$$\sim \frac{4\pi}{k^2}(kR(kR+1))\frac{1}{2} \tag{1.36}$$
$$\sim 2\pi R^2 \tag{1.37}$$

Now the total cross section is only twice the area of the sphere. What goes on?

Let's consider the scattering amplitude again

$$f_l(\theta) = \frac{e^{2i\delta_l} - 1}{2ik} \equiv f_{l,scat} + f_{l,0} \tag{1.38}$$

where $f_{l,scat}$ is the part that is scattered off the sphere, and $f_{l,0}$ is the piece of the outgoing wave that was there in the first place. We have that

$$\begin{aligned}
\int |f_{scat}|^2 d\Omega &= \int \left|\sum_{l=0}^{\infty}(2l+1)f_{l,scat}P_l(\theta)\right|^2 d\Omega \\
&= \int \left|\sum_{l=0}^{\infty}\sum_{l'=0}^{\infty}(2l+1)(2l'+1)\sqrt{\frac{4\pi}{(2l+1)}}\sqrt{\frac{4\pi}{(2l'+1)}} f_{l,scat}Y_{l0}(\theta) f_{l',scat}Y^*_{l'0}(\theta)\right|^2 d\Omega \\
&= 4\pi \sum_{l=0}^{l=Rk}(2l+1)|\frac{f_{l,scat}}{2ik}|^2 \\
&= \frac{\pi(kR)^2}{k^2} = \pi R^2
\end{aligned}$$