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Lecture XI

1.1.1 Hard Sphere scattering

The probability of finding the particle inside the radius R of the sphere is zero. It is zero at the boundary. So

$$\psi^+(R) = j_l(kR)\cos\delta_l - n_l(kR)\sin\delta_l = 0 \tag{1.1}$$

$$j_l(kR)\cos\delta_l = n_l(kR)\sin\delta_l \to \tan\delta_l = \frac{j_l(kR)}{n_l(kR)} \tag{1.2}$$

As we have determined all of the phase shifts, we can compute the exact differential cross section. Let's consider the limits. Suppose that the energy of the incoming particle is very low. So low that the angular momentum $\hbar kR \ll \hbar$. Then only the $l = 0$ partial wave will contribute. Partial waves with $l > 0$ have zero amplitude of appearing at R. Also consider the small x limit of $j_l(x), n_l(x)$.

$$\begin{aligned}
\lim_{x\to 0} j_l(x) &= \frac{2^l l!}{(2l+1)!}x^l \\
\lim_{x\to 0} n_l(x) &= -\frac{(2l)!}{2^l l!}\frac{1}{x^{l+1}} \\
\to \lim_{kR\to 0} \tan\delta_l &\sim \frac{(2^l l!)^2}{(2l+1)(2l)!}x^{(2l+1)}
\end{aligned}$$

Clearly the $l = 0$ partial wave is the most important component of the wave function when $kR \ll 1$.

The $l = 0$ phase shift is

$$\tan\delta_0 = \frac{\sin kR/kR}{\cos kR/KR} = -\tan kR \to \delta_0 = kR \tag{1.3}$$

The differential cross section for s-wave scattering is

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = |\frac{1}{k}\sin kR|^2 \sim R^2 \tag{1.4}$$

No angular dependence of course. The total cross section is $\sigma_t = 4\pi R^2$, four times the geometric cross section.

In the limit of high energy, where $kR \gg 1$, we can use the asymptotic form for j_l and n_l. Then all partial waves with $l \le kR$ will contribute.

$$\lim_{kR\to\infty} j_l(kR) = \frac{\sin(kR - l\frac{\pi}{2})}{kR} \tag{1.5}$$

$$\lim_{kR\to\infty} n_l(kR) = -\frac{\cos(kR - l\frac{\pi}{2})}{kR} \tag{1.6}$$

$$\tag{1.7}$$

Then

$$\lim_{kR\to\infty} \tan\delta_l = -\tan(kR - l\frac{\pi}{2}) \tag{1.8}$$

$$\sin^2\delta_l = \sin^2(kR - l\frac{\pi}{2}) \tag{1.9}$$

$$\int d\Omega |f(\theta)|^2 = \frac{4\pi}{k^2}\sum_l (2l+1)\sin^2(kR - l\frac{\pi}{2}) \tag{1.10}$$

So let's sum to kR.

$$\int d\Omega |f(\theta)|^2 = \frac{4\pi}{k^2}\sum_{l=0}^{kR} (2l+1)\sin^2(kR - l\frac{\pi}{2}) \tag{1.11}$$

$$\sim \frac{4\pi}{k^2}(kR(kR+1))\frac{1}{2} \tag{1.12}$$

$$\sim 2\pi R^2 \tag{1.13}$$

Now the total cross section is only twice the area of the sphere. What goes on?

Let's consider the scattering amplitude again

$$f_l(\theta) = \frac{e^{2i\delta_l} - 1}{2ik} \equiv f_{l,scat} + f_{l,0} \tag{1.14}$$

where $f_{l,scat}$ is the part that is scattered off the sphere, and $f_{l,0}$ is the piece of the outgoing wave that was there in the first place. We have that

$$\begin{aligned}
\int |f_{scat}|^2 d\Omega &= \int \left| \sum_{l=0}^{\infty} (2l+1) f_{l,scat} P_l(\theta) \right|^2 d\Omega \\
&= \int \left| \sum_{l=0}^{\infty}\sum_{l'=0}^{\infty} (2l+1)(2l'+1)\sqrt{\frac{4\pi}{(2l+1)}}\sqrt{\frac{4\pi}{(2l'+1)}} f_{l,scat} Y_{l0}(\theta) f_{l',scat} Y^*_{l'0}(\theta) \right|^2 d\Omega \\
&= 4\pi \sum_{l=0}^{l=Rk} (2l+1) |\frac{f_{l,scat}}{2ik}|^2 \\
&= \frac{\pi(kR)^2}{k^2} = \pi R^2
\end{aligned}$$

1.2 Spin dependent scattering

We begin by generalizing the transition operator to include spin. Then

$$\langle \mathbf{k}' \mid T \mid \mathbf{k} \rangle \rightarrow \langle \mathbf{k}', \nu_f \mid T \mid \mathbf{k}, \nu_i \rangle \tag{1.15}$$

The scattered state is

$$\psi^+ = \frac{1}{(2\pi)^{3/2}} \left(e^{i(\mathbf{k_i} \cdot \mathbf{r})} \mid \nu_i \rangle + \frac{e^{ikr}}{r} \sum_{\nu_f} \mid \nu_f \rangle f(\mathbf{k}_f \nu_f; \mathbf{k}_i \nu_i) \right),$$

so that

$$\frac{d\sigma_{i\rightarrow f}}{d\Omega} = |f(\mathbf{k}_f \nu_f | \mathbf{k}_i \nu_i)|^2.$$

The transition operator now acts in spin space as well as momentum space and we can write

$$\langle \mathbf{k}', \nu_f \mid T \mid \mathbf{k}, \nu_i \rangle \rightarrow \langle \nu_f \mid M(\mathbf{k}', \mathbf{k}) \mid \nu_i \rangle \tag{1.16}$$

Let's construct an operator M that is invariant with respect to parity and time reversal, and rotations. According to that last requirement it is a scalar. If we suppose that initial and final states have spin $\frac{1}{2}$ and the scattering potential is spherically symmetric, (to keep it simple), then it is a 2 x 2 Hermitian operator. The most general such operator is constructed as a linear combination of the identity matrix and the pauli matrices. The available kinematic parameters are the initial and final momenta and spin. The scalar combinations are

$$(\mathbf{k}' \times \mathbf{k}) \cdot \sigma, \quad (\mathbf{k}' - \mathbf{k}) \cdot \sigma, \quad (\mathbf{k}' + \mathbf{k}) \cdot \sigma$$

The first will be a scalar and the second and third pseudo-scalar, (since spin is a pseudo vector).

The most general transition scattering operator has the form

$$M(\mathbf{k}', \mathbf{k}) = g_1 + g_2(\mathbf{k}' \times \mathbf{k}) \cdot \sigma + g_3(\mathbf{k}' - \mathbf{k}) \cdot \sigma + g_4(\mathbf{k}' + \mathbf{k}) \cdot \sigma \tag{1.17}$$

The third and fourth terms are not invariant with respect to parity as the momentum vector changes sign but the angular momentum vector does not. Only the first and second terms are allowed for a parity conserving process. Note also that the third term is not time reversal invariant. Time reversal changes the sign of both momentum and spin and $k_i \rightarrow k_f$. So the most general form for the scattering operator that is rotationally invariant and preserves time and space reversal symmetry is

$$M(\mathbf{k}_f, \mathbf{k}_i) = g_1(k, \theta) + g_2(k, \theta)\sigma \cdot \frac{\mathbf{k}' \times \mathbf{k}}{|\mathbf{k}' \times \mathbf{k}|} = g_1(k, \theta) + g_2(k, \theta)\sigma \cdot \hat{\mathbf{n}} \tag{1.18}$$

where $\hat{\mathbf{n}}$ is the unit vector perpendicular to the scattering plane. Of course Equation 1.42 is the most general form of the scattering operator. The scattering potential might not interact with the particle spin and then we would have that $g_2 = 0$. But the potential is presumed rotationally invariant, and invariant under reflection so that if g_2 is not zero, then the most general form for the potential is

$$V = V_0(r) + \sigma \cdot \mathbf{L} V_1(r) \tag{1.19}$$

$\mathbf{L}$ being the only available axial vector in the space of the spatial coordinate (and parallel to $\hat{\mathbf{n}}$).