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Lecture XII

1.2 Spin dependent scattering

We begin by generalizing the transition operator to include spin. Then

$$\langle \mathbf{k}' \mid T \mid \mathbf{k} \rangle \rightarrow \langle \mathbf{k}', \nu_f \mid T \mid \mathbf{k}, \nu_i \rangle \tag{1.1}$$

The scattered state is

$$\psi^+ = \frac{1}{(2\pi)^{3/2}} \left(e^{i(\mathbf{k_i}\cdot\mathbf{r})} \mid \nu_i \rangle + \frac{e^{ikr}}{r} \sum_{\nu_f} \mid \nu_f \rangle f(\mathbf{k}_f \nu_f; \mathbf{k}_i \nu_i) \right),$$

so that

$$\frac{d\sigma_{i\rightarrow f}}{d\Omega} = |f(\mathbf{k}_f \nu_f | \mathbf{k}_i \nu_i)|^2.$$

The transition operator now acts in spin space as well as momentum space and we can write

$$\langle \mathbf{k}', \nu_f \mid T \mid \mathbf{k}, \nu_i \rangle \rightarrow \langle \nu_f \mid M(\mathbf{k}', \mathbf{k}) \mid \nu_i \rangle \tag{1.2}$$

Let's construct an operator M that is invariant with respect to parity and time reversal, and rotations. According to that last requirement it is a scalar. If we suppose that initial and final states have spin $\frac{1}{2}$ and the scattering potential is spherically symmetric, (to keep it simple), then it is a 2 x 2 Hermitian operator. The most general such operator is constructed as a linear combination of the identity matrix and the pauli matrices. The available kinematic parameters are the initial and final momenta and spin. The scalar combinations are

$$(\mathbf{k}' \times \mathbf{k}) \cdot \sigma, \quad (\mathbf{k}' - \mathbf{k}) \cdot \sigma, \quad (\mathbf{k}' + \mathbf{k}) \cdot \sigma$$

The first will be a scalar and the second and third pseudo-scalar, (since spin is a pseudo vector).

The most general transition scattering operator has the form

$$M(\mathbf{k}', \mathbf{k}) = g_1 + g_2(\mathbf{k}' \times \mathbf{k}) \cdot \sigma + g_3(\mathbf{k}' - \mathbf{k}) \cdot \sigma + g_4(\mathbf{k}' + \mathbf{k}) \cdot \sigma \tag{1.3}$$

The third and fourth terms are not invariant with respect to parity as the momentum vector changes sign but the angular momentum vector does not. Only the first and second terms are allowed for a parity conserving process. Note also that the third term is not time reversal invariant. Time reversal changes the sign of both momentum and spin and $k_i \rightarrow k_f$. So the most general form for the scattering operator that is rotationally invariant and preserves time and space reversal symmetry is

$$M(\mathbf{k}_f, \mathbf{k}_i) = g_1(k, \theta) + g_2(k, \theta) \sigma \cdot \frac{\mathbf{k}' \times \mathbf{k}}{|\mathbf{k}' \times \mathbf{k}|} = g_1(k, \theta) + g_2(k, \theta) \sigma \cdot \hat{\mathbf{n}} \tag{1.4}$$

where $\hat{\mathbf{n}}$ is the unit vector perpendicular to the scattering plane. Of course Equation 1.4 is the most general form of the scattering operator. The scattering potential might not interact with the particle spin and then we would have that $g_2 = 0$. But the potential is presumed rotationally invariant, and invariant under reflection so that if g_2 is not zero, then the most general form for the potential is

$$V = V_0(r) + \sigma \cdot \mathbf{L} V_1(r) \tag{1.5}$$

$\mathbf{L}$ being the only available axial vector in the space of the spatial coordinate (and parallel to $\hat{\mathbf{n}}$).

1.3 Symmetry

We know that if the Hamiltonian is invariant with respect to translation, that it commutes with the generator of translation, namely momentum and momentum is conserved, since by Heisenberg equations of motion

$$\frac{dG}{dt} = \frac{1}{i\hbar}[G, H]$$

For free particle, translation invariant and momentum is conserved. For central potential, rotation invariant, and angular momentum is conserved, etc.

1.3.1 Parity

Now let's talk about the discrete symmetry, parity or space inversion, turning right handed into left handed. Typically, the mirror image of a physical process is indistinguishable from the real thing. The reflection of a collision of billiard balls is governed by the same Newton's laws as the real image. Maxwell's equations, and the Lorentz force law are invariant with respect to their inverted version. We define a parity operator π and the inverted state is

$$|\,\alpha\rangle \rightarrow \pi|\,\alpha\rangle$$

The expectation value of

$$\langle \mathbf{x} \rangle \rightarrow -\langle \mathbf{x} \rangle \rightarrow \left\langle \alpha \mid \pi^\dagger \mathbf{x} \pi \mid \alpha \right\rangle = -\left\langle \alpha \mid \mathbf{x} \mid \alpha \right\rangle$$

from which we learn that

$$\pi^\dagger \mathbf{x} \pi = -\mathbf{x} \rightarrow \mathbf{x}\pi = -\pi\mathbf{x}$$

What is the effect on eigenkets?

$$\pi\mathbf{x}|\,\mathbf{x}'\rangle = -\mathbf{x}\pi|\,\mathbf{x}'\rangle \rightarrow \mathbf{x}'\pi|\,\mathbf{x}'\rangle = -\mathbf{x}\pi|\,\mathbf{x}'\rangle$$

so that $\pi|\,\mathbf{x}'\rangle$ is an eigenket of $\mathbf{x}$ with eigenvalue $-\mathbf{x}'$ and

$$\pi|\,\mathbf{x}'\rangle = e^{i\delta}|\,-\mathbf{x}'\rangle$$

Then

$$\pi^2|\,\mathbf{x}'\rangle = e^{2i\delta}|\,\mathbf{x}'\rangle = |\,\mathbf{x}'\rangle$$

so it is reasonable to set $\delta = 0$.

Parity eigenkets

Supose $\pi|\,\alpha\rangle = \lambda|\,\alpha\rangle$. Then

$$\langle \alpha \mid x \mid \alpha \rangle = -\left\langle \alpha \mid \pi^\dagger x \pi \mid \alpha \right\rangle = -|\lambda|^2 \left\langle \alpha \mid x \mid \alpha \right\rangle \Rightarrow \langle \alpha \mid x \mid \alpha \rangle = 0$$

The expectation value of $\langle x \rangle$ for states of definite parity is zero. Also

$$\pi^2|\,\alpha\rangle = \lambda^2|\,\alpha\rangle = |\,\alpha\rangle \Rightarrow \lambda = \pm 1$$

The eigenvalues of the parity operator are ± 1.

Momentum

Momentum operator is like position, time derivative, so odd with respect to parity. We expect

$$\left\langle \alpha \mid \pi^{\dagger} p \pi \mid \alpha \right\rangle = -\left\langle \alpha \mid p \mid \alpha \right\rangle$$

Or we can derive this relationship from the definition $\{x, \pi\} = 0$. We note that translation followed by space inversion is equivalent to inversion and then translation in the opposite direction.

$$\begin{aligned} \mathcal{T}(d\mathbf{x})\pi &= \pi\mathcal{T}(-d\mathbf{x}) \\ \pi^{\dagger}(1 - \frac{i}{\hbar}\mathbf{p}\cdot\mathbf{dx})\pi &= (1 + \frac{i}{\hbar}\mathbf{p}\cdot d\mathbf{x}) \\ \rightarrow \pi^{\dagger}\mathbf{p}\pi &= -\mathbf{p} \end{aligned}$$

Angular momentum

For orbital angular momentum

$$\pi^{\dagger} L_k \pi = \epsilon_{ijk}\pi^{\dagger} r_i p_j \pi = \epsilon_{ijk}(-r_i)(-p_j) = \epsilon_{ijk}(r_i)(p_j) = L_k$$

and we get that $[\pi, \mathbf{L}] = 0$. Also, angular momentum is the generator of rotations and the the space inversion matrix

$$S_{inv} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

commutes with the rotation matrix. That is, we can rotate and then invert or invert and then rotate and end up in the same place. Since rotation and inversion commute so do angular momentum and the inversion operator. Also space inversion commutes with $\mathcal{D}$ and therefore with the generator of the rotations. And also, look in the mirror and see what happens.

Note that inversion in x, y, and z is equivalent to reflection in one plane and then 180 deg rotation about the axis perpendicular to that plane. For example we accomplish reflection in the y-z plane with the help of

$$S_{yz} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Rotation about the x axis is given by

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \rightarrow_{\theta=\pi} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and we see that

$$S_{yz} R_x(\pi) = S_{inv}$$

Remember $SO(3)$ are orthogonal with determinant 1.

1.3.2 Wave functions

Consider

$$| \alpha\rangle = \int dx' | x'\rangle\langle x' | \alpha\rangle$$

Then

$$\begin{aligned}
\pi| \alpha\rangle &= \int dx' \pi | x'\rangle\langle x' | \alpha\rangle \\
\pi| \alpha\rangle &= \int dx' | -x'\rangle\langle x' | \alpha\rangle \\
\pi| \alpha\rangle &= \int dx' | x'\rangle\langle -x' | \alpha\rangle
\end{aligned}$$

Now if $| \alpha\rangle$ is a parity eigenket then

$$\pi| \alpha\rangle = \lambda| \alpha\rangle = \int dx' \lambda | x'\rangle\langle x' | \alpha\rangle$$

and then on comparison with that last equation we have that

$$\lambda\langle x' | \alpha\rangle = \langle -x' | \alpha\rangle \rightarrow \lambda\psi(x) = \psi(-x)$$

But $\lambda = \pm 1$ so

$$\psi(x) \rightarrow \pm\psi(-x)$$

Some states are parity eigenstates and some are not. Momentum no, although we could easily write them as linear combinations. The momentum operator does not commute with π. But the angular momentum operator does, so eigenstates of L are eigenstates of π as long as there is no degeneracy. So consider $\psi_{\alpha,l,m} = R_\alpha(r)Y_{lm}$ Under reflection

$$r \rightarrow r, \quad \theta \rightarrow \pi - \theta, \quad \phi \rightarrow \phi + \pi$$

To figure out the parity of a spherical harmonic consider

$$Y_{l,l} = \sin^l \theta e^{il\phi} \rightarrow \sin^l(\pi - \theta)e^{il(\pi+\phi)} = \sin^l \theta(-1)^l e^{il\phi} = (-1)^l Y_{l,l}$$

All of the $Y_{l,m}$ are connect to the $Y_{l,l}$ state by L_- which is constructed of angular momentum operators which all commute with π so all of the substates have the same parity.

Energy eigenkets

Now if the hamiltonian is invariant with respect to parity, and the eigenkets are non degenerate, then eigenkets of energy have definite parity. First

$$\pi H| n\rangle = H\pi| n\rangle = E_n\pi| n\rangle$$

so we see that the space inverted version of an energy eigenket is also an energy eigenket with the same energy. It is also the same state if there is no degeneracy. But that does not show that it is a parity eigenket. (Example of 2s and 1p). But we can construct a parity eigenket

$$\frac{1}{2}(1 \pm \pi)|\, n\rangle$$

This state has energy eigenvalue E_n and

$$\pi\frac{1}{2}(1 \pm \pi)|\, n\rangle = \frac{1}{2}(\pi \pm 1)|\, n\rangle = \pm\frac{1}{2}(1 \pm \pi)|\, n\rangle$$

it is an eigenket of π with eigenvalues ± 1 and it must be the same state as $|\, n\rangle$ since there is assumed no degeneracy.

Reflections

Let's examine Maxwell's equations on inversion. Suppose I have a positively charged plate at $z = z_{high}$ and negatively charged plate at $z = z_{low}$, both parallel to the xy plane. There will be an electric field that points up. Reflection in the x-y plane will put the negatively charged plate below the positively charged plate so the electric field will point down. Evidently $E \rightarrow -E$.

Now imagine that I have a current carrying wire with current flowing up. Then using the right hand rule I get a magnetic field counterclockwise about the wire (when looking from above). In the image reflected in the x-y plane I see the current flowing down. My right hand now looks like a left hand, so when I use it to determine the direction of the field I find that it is clounterclockwise (when looking down) as before. We find that $B \rightarrow B$.

The effect of the curl is to change the sign. To see this imagine a plane wave

$$\mathbf{E} = \hat{\epsilon} E_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

where the direction of $\mathbf{k} = \mathbf{p}/\hbar$ is the direction in which the wave is propagating and whose sign is changed on inversion. Then

$$\nabla \times \mathbf{E} = i(\mathbf{k} \times \hat{\epsilon}) E_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

We see that taking the curl is equivalent to taking the cross product with the momentum vector. For that matter, the divergenge

$$\nabla \cdot \mathbf{E} = i\mathbf{k} \cdot \mathbf{E}$$

also changes sign. So

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

Everything is OK. Also

$$H = \frac{1}{2m}(\mathbf{p} - \frac{e}{c}\mathbf{A})^2 + e\phi$$

$\mathbf{p}$ and $\mathbf{A}$ both change signs. No change in the potential ϕ. (Consider a point charge at $z = z_{up}$. The potential at point above z_{up} on the z axis will be the same as at that point reflected to below $-z_{up}$).

And the Lorentz force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = m\frac{d^2\mathbf{x}}{dt^2}$$

(Acceleration obviously changes sign. Imagine a charge between those two charged plates)

1.3.3 Λ_0 decay

Production: $p+\pi^- \to \Lambda_0 + K_0$ Decay $\Lambda_0 \to \pi_- + p$ In the rest frame of the Λ_0 with the Λ_0 spin $+\frac{1}{2}$ along the z-direction there is some amplitude a that the proton will head up with spin $+\frac{1}{2}$ and the pion down to conserve momentum. And if the Λ_0 is spin down there is some amplitude b that the proton will head up with spin down. There is zero amplitude for either of the above with proton spin reversed since angular momentum is not conserved and note that there is no z-component of orbital angular momentum because the momentum is in the z-direction. The total probability or a proton along the +z-direction is

$$P_{tot}(+z) = P_+ + P_- = |a\left\langle \frac{1}{2}, +\frac{1}{2} \mid \Lambda_0 \right\rangle|^2 + |b\left\langle \frac{1}{2}, -\frac{1}{2} \mid \Lambda_0 \right\rangle|^2$$

Now suppose the Λ_0 are polarized and we know that all have $+\frac{1}{2}$, then what is the probability that the proton will be emitted in the z' direction?

The amplitude for spin $+\frac{1}{2}$ in the z' direction is

$$d^{\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}}(\theta) = \cos\theta/2$$

and the amplitude for spin $-\frac{1}{2}$ is

$$d^{\frac{1}{2}}_{-\frac{1}{2},\frac{1}{2}}(\theta) = \sin\theta/2$$

The

$$\begin{aligned}
P_{tot}(+z') &= |a\cos\theta/2|^2 + |b\sin\theta/2|^2 \\
&= |b|^2 - (|b|^2 - |a|^2)\cos^2\theta/2 = b^2 - (b^2 - a^2)\frac{1}{2}(1+\cos\theta) \\
&= \frac{1}{2}\left((b^2+a^2) - (b^2-a^2)\cos\theta\right) \\
&= \frac{1}{2}(b^2+a^2)\left(1 - \frac{b^2-a^2}{b^2+a^2}\cos\theta\right) \\
&= N(1-\alpha\cos\theta)
\end{aligned}$$

If $|a| = |b|$ (parity is conserved), then the distribution is isotropic.

Alternatively we can consider more generally the implications of the conservation of angular momentum for the final state wave function. We suppose that in the initial state, the Λ_0 has spin $j = +\frac{1}{2}$ and $j_z = +\frac{1}{2}$. Then of course the final state has the same quantum numbers. But in the final state there is orbital as well as spin angular momentum. The most general form of the final state wave function is

$$\mid \psi\rangle = \mid \frac{1}{2}, \frac{1}{2}\rangle \quad = \quad a_p(c_1 \mid 1,1\rangle \mid \frac{1}{2}, -\frac{1}{2}\rangle + c_2 \mid 1,0\rangle \mid \frac{1}{2}\frac{1}{2}\rangle) + a_s(\mid 0,0\rangle \mid \frac{1}{2}, \frac{1}{2}\rangle$$

c_1 and c_2 are CB coefficients.

$$\begin{aligned}
c_1 &= \left\langle 1, \frac{1}{2}; 1, -\frac{1}{2} \mid 1, \frac{1}{2}; \frac{1}{2}, \frac{1}{2} \right\rangle = \sqrt{\frac{2}{3}} \\
c_2 &= \left\langle 1, \frac{1}{2}; 0, \frac{1}{2} \mid 1, \frac{1}{2}; \frac{1}{2}, \frac{1}{2} \right\rangle = -\sqrt{\frac{1}{3}}
\end{aligned}$$

Then

$$\begin{aligned}
\psi &= a_p(\sqrt{\frac{2}{3}}Y_1^1(\theta,\phi)|\ \frac{1}{2},-\frac{1}{2}\rangle - \sqrt{\frac{1}{3}}Y_1^0(\theta,\phi)|\ \frac{1}{2},\frac{1}{2}\rangle) + a_s Y_0^0|\ \frac{1}{2},\frac{1}{2}\rangle \\
|\psi|^2 &= |a_p|^2\left(\frac{2}{3}|Y_1^1|^2 + \frac{1}{3}|Y_1^0|^2)\right) + |a_s|^2|Y_0^0|^2 - (a_p^* a_s + a_s^* a_p)\sqrt{\frac{1}{3}}Y_1^0 Y_0^0 \\
&= \frac{1}{4\pi}\left(|a_p|^2(\sin^2\theta + \cos^2\theta) + |a_s^2| - (a_p^* a_s + a_s a_p^*)\cos\theta\right) \\
&= \frac{1}{4\pi}\left(|a_p|^2 + |a_s|^2 - (a_p^* a_s + a_s a_p^*)\cos\theta\right) \\
&\sim 1 - \alpha\cos\theta
\end{aligned}$$