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Lecture XIII

1.1.1 Λ_0 decay

Production: $p+\pi^- \rightarrow \Lambda_0 + K_0$ Decay $\Lambda_0 \rightarrow \pi_- + p$ In the rest frame of the Λ_0 with the Λ_0 spin $+\frac{1}{2}$ along the z-direction there is some amplitude a that the proton will head up with spin $+\frac{1}{2}$ and the pion down to conserve momentum. And if the Λ_0 is spin down there is some amplitude b that the proton will head up with spin down. There is zero amplitude for either of the above with proton spin reversed since angular momentum is not conserved and note that there is no z-component of orbital angular momentum because the momentum is in the z-direction. The total probability or a proton along the +z-direction is

$$P_{tot}(+z) = P_+ + P_- = |a\left\langle \frac{1}{2}, +\frac{1}{2} \mid \Lambda_0 \right\rangle|^2 + |b\left\langle \frac{1}{2}, -\frac{1}{2} \mid \Lambda_0 \right\rangle|^2$$

Now suppose the Λ_0 are polarized and we know that all have $+\frac{1}{2}$, then what is the probability that the proton will be emitted in the z' direction?

The amplitude for spin $+\frac{1}{2}$ in the z' direction is

$$d^{\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}}(\theta) = \cos\theta/2$$

and the amplitude for spin $-\frac{1}{2}$ is

$$d^{\frac{1}{2}}_{-\frac{1}{2},\frac{1}{2}}(\theta) = \sin\theta/2$$

The

$$\begin{aligned}
P_{tot}(+z') &= |a\cos\theta/2|^2 + |b\sin\theta/2|^2 \\
&= |b|^2 - (|b|^2 - |a|^2)\cos^2\theta/2 = b^2 - (b^2 - a^2)\frac{1}{2}(1+\cos\theta) \\
&= \frac{1}{2}\left((b^2+a^2) - (b^2-a^2)\cos\theta\right) \\
&= \frac{1}{2}(b^2+a^2)\left(1 - \frac{b^2-a^2}{b^2+a^2}\cos\theta\right) \\
&= N(1-\alpha\cos\theta)
\end{aligned}$$

If $|a| = |b|$ (parity is conserved), then the distribution is isotropic.

Alternatively we can consider more generally the implications of the conservation of angular momentum for the final state wave function. We suppose that in the initial state, the Λ_0 has spin $j = +\frac{1}{2}$ and $j_z = +\frac{1}{2}$. Then of course the final state has the same quantum numbers. But in the final state there is orbital as well as spin angular momentum. The most general form of the final state wave function is

$$|\psi\rangle = |\frac{1}{2}, \frac{1}{2}\rangle = a_p(c_1|1,1\rangle|\frac{1}{2}, -\frac{1}{2}\rangle + c_2|1,0\rangle|\frac{1}{2}\frac{1}{2}\rangle) + a_s(|0,0\rangle|\frac{1}{2}, \frac{1}{2}\rangle$$

c_1 and c_2 are CB coefficients.

$$\begin{aligned}
c_1 &= \left\langle 1, \frac{1}{2}; 1, -\frac{1}{2} \mid 1, \frac{1}{2}; \frac{1}{2}, \frac{1}{2} \right\rangle = \sqrt{\frac{2}{3}} \\
c_2 &= \left\langle 1, \frac{1}{2}; 0, \frac{1}{2} \mid 1, \frac{1}{2}; \frac{1}{2}, \frac{1}{2} \right\rangle = -\sqrt{\frac{1}{3}}
\end{aligned}$$

Then

$$\begin{aligned}
\psi &= a_p(\sqrt{\frac{2}{3}} Y_1^1(\theta,\phi) | \frac{1}{2}, -\frac{1}{2}\rangle - \sqrt{\frac{1}{3}} Y_1^0(\theta,\phi) | \frac{1}{2}, \frac{1}{2}\rangle) + a_s Y_0^0 | \frac{1}{2}, \frac{1}{2}\rangle \\
|\psi|^2 &= |a_p|^2 \left(\frac{2}{3}|Y_1^1|^2 + \frac{1}{3}|Y_1^0|^2) \right) + |a_s|^2 |Y_0^0|^2 - (a_p^* a_s + a_s^* a_p) \sqrt{\frac{1}{3}} Y_1^0 Y_0^0 \\
&= \frac{1}{4\pi} \left(|a_p|^2 (\sin^2\theta + \cos^2\theta) + |a_s^2| - (a_p^* a_s + a_s a_p^*) \cos\theta \right) \\
&= \frac{1}{4\pi} \left(|a_p|^2 + |a_s|^2 - (a_p^* a_s + a_s a_p^*) \cos\theta \right) \\
&\sim 1 - \alpha \cos\theta
\end{aligned}$$

1.1.2 Time reversal

Time reversal is a tricky business. First consider a classical example. Suppose a planet is in orbit about the sun. At some time say $t = 0$ it has a velocity $\mathbf{v}(0)$. If we reverse the velocity so that $\mathbf{v}(0) \rightarrow -\mathbf{v}(0)$ then the planet will retrace it's trajectory so that $x_R(t) = x(-t)$ and $\mathbf{v}_R(t) = -\mathbf{v}(-t)$ where $x_R(t)$ is the position of the reversed planet at t and $\mathbf{v}_R(t)$ is the velocity of the reversed planet at time t. So we reverse the state and propagate by t and we should end up with exactly the same position and velocity as if we had propagated the original state by $-t$ and then reversed the velocity.

Now quantum mechanics. We have a state $|\,\psi\rangle$ and $|\,\psi^R\rangle$ the reversed state and suppose we have some operator that effects the reversal so that

$$I_t|\,\psi\rangle = |\,\psi^R\rangle$$

If $|\,\psi\rangle = |\,\mathbf{p}'\rangle$ represents a state with definite momentum then $|\,\psi^R\rangle = |\,-\mathbf{p}'\rangle$ and we expect that

$$\mathbf{p}|\,\psi^R\rangle = -\mathbf{p}'|\,\psi^R\rangle$$

and

$$\left\langle\psi^R \mid \mathbf{p} \mid \psi^R\right\rangle = -\left\langle\psi \mid \mathbf{p} \mid \psi\right\rangle \rightarrow I_t^{-1}\mathbf{p}I_t = -\mathbf{p}$$

So far so good. The reversed state at time t, $|\,\psi(t)^R\rangle$, should be the same as the original state at $-t$ and then reversed. So we have the state at $t = 0$ and we reverse it and propagate it by an infinitesimal time δt. Meanwhile we propagate the original state by $-\delta t$ and then reverse it and we should end up in the same place. That is

$$(1 - \frac{i}{\hbar}H\delta t)I_t|\,\psi\rangle = I_t(1 + \frac{i}{\hbar}H\delta)|\,\psi\rangle \rightarrow iHI_t = -I_t iH \tag{1.1}$$

Unitary and anti unitary operators

The parity operator π, like all of the others that we have discussed, is unitary. We found that $\pi^2|\,\alpha\rangle = |\,\alpha\rangle$ so we see that $\pi\pi = 1$. Therefore $\pi = \pi^{-1}$ and $\pi = \pi^\dagger$. Finally $\left\langle\alpha \mid \pi^{-1}\pi \mid \beta\right\rangle = \langle\alpha \mid \beta\rangle$. The parity operator preserves the inner product.

We like unitary operators so that the inner product is invariant. In particular, if U is unitary then $U|\,\alpha\rangle = |\,\alpha\rangle'$ and $U|\,\beta\rangle = |\,\beta\rangle'$ and

$$\langle\alpha' \mid \beta'\rangle = \left\langle\alpha \mid U^\dagger U \mid \beta\right\rangle = \langle\alpha \mid \beta\rangle$$

which is nice but perhaps not essential. What is essential is that

$$|\langle\alpha' \mid \beta'\rangle| = |\langle\alpha \mid \beta\rangle|.$$

It turns out that this transformation is necessarily unitary or anti-unitary. Those are the only possibilities. As we will see in a moment a unitary operator does not work for time reversal. We consider an anti-unitary operator I_t such that $I_t|\,\alpha\rangle = |\,\alpha'\rangle$ and $I_t|\,\beta\rangle = |\,\beta'\rangle$ but now

$$\langle\alpha' \mid \beta'\rangle = \langle\alpha \mid \beta\rangle^*$$

so we preserve length but perhaps not phase. In practice if

$$|\,\psi\rangle = a_1|\,1\rangle + a_2|\,2\rangle$$

then

$$I_t|\,\psi\rangle = a_1^* I_t|\,1\rangle + a_2^* I_t|\,2\rangle$$

In general we can write an anti-unitary operator as the product of a unitary operator and the complex conjugation operator. That is

$$I_t = UK, \qquad I_t^{-1} = KU^\dagger$$

where U is unitary and K takes the complex conjugate of the coefficients. The anti-unitary transformation is not useful as a member of a continuous group because applying I_t twice results in a unitary transformation and we like the composition rule where we can apply two transformations to get a third. The only candidates for an anti-unitary transformation are where T^2 gets you back where you started, namely parity and time reversal, charge conjugation, interchange.

Transformation of H under time reversal

Now back to the problem at hand. Referring back to Equation 1.1, if I_t is unitary then we conclude that $HI_t = -I_tH$ and for some eigenket of H

$$I_tH|\,n\rangle = I_tE_n|\,n\rangle = E_nI_t|\,n\rangle = -I_tH|\,n\rangle = -E_nI_t|\,n\rangle$$

which says that we have an eigenket of H with negative energy. That makes no sense. If we have a free particle and turn it around, we don't get negative energy since energy scales as p^2. The other choice is that I_t is anti-unitary. Then Equation 1.1 gives us

$$HI_t = I_tH \rightarrow [H, I_t] = 0$$

Expectation value of anit-unitary operator

Suppose that

$$|\,\tilde{\alpha}\rangle = I_t|\,\alpha\rangle, \quad |\,\tilde{\beta}\rangle = I_t|\,\beta\rangle$$

and I_t is anti-unitary. Then

$$\left\langle \tilde{\beta} \mid \tilde{\alpha} \right\rangle = \langle \alpha \mid \beta \rangle$$

and if $|\,\gamma\rangle = A^\dagger|\,\beta\rangle$ then

$$\langle \beta \mid A \mid \alpha \rangle = \langle \gamma \mid \alpha \rangle = \langle \tilde{\alpha} \mid \tilde{\gamma} \rangle = \left\langle \tilde{\alpha} \mid I_tA^\dagger \mid \beta \right\rangle = \left\langle \tilde{\alpha} \mid I_tA^\dagger I_t^{-1} I_t \mid \beta \right\rangle = \left\langle \tilde{\alpha} \mid I_tA^\dagger I_t^{-1} \mid \tilde{\beta} \right\rangle$$

So the rule for the anit-unitary operator I_t is that for any linear operator A,

$$\langle \beta \mid A \mid \alpha \rangle = \left\langle \tilde{\alpha} \mid I_tA^\dagger I_t^{-1} \mid \tilde{\beta} \right\rangle$$

If A is Hermitian then

$$\langle \beta \mid A \mid \alpha \rangle = \left\langle \tilde{\alpha} \mid I_tAI_t^{-1} \mid \tilde{\beta} \right\rangle$$

An observable is even or odd if

$$I_tAI_t^{-1} = \pm A.$$

Therefore

$$\langle \alpha \mid A \mid \alpha \rangle = \pm \left\langle \tilde{\alpha} \mid I_tAI_t^{-1} \mid \tilde{\alpha} \right\rangle$$

Transformation of position and angular momentum

The expectation value of position does not change under time reversal so

$$I_t^{-1}\mathbf{x}I_t = \mathbf{x}$$

In order that time reversal operation to effect no geometrical transformation it must be that angular momentum is odd. Then

$$I_t^{-1}\mathcal{D}I_t = I_t^{-1}e^{-\frac{i}{\hbar}\mathbf{J}\cdot\phi}I_t = e^{\frac{i}{\hbar}I_t^{-1}\mathbf{J}\cdot\phi I_t} = \mathcal{D}$$

as long as $I_t^{-1}\mathbf{J}I_t = -\mathbf{J}$ and this is also in agreement with the notion that x is even and p is odd.

Uncertainty principle

In view of the above

$$\begin{aligned}
[x_i, p_j] &= i\hbar\delta_{ij} \\
I_t[x_i, p_j]I_t^{-1} &= I_t i\hbar\delta_{ij}I_t^{-1} \\
[x_i, -p_j] &= -i\hbar\delta_{ij}
\end{aligned}$$

so the uncertainty principle hangs together. Note that $I_t\mathbf{J}I_t^{-1} = -\mathbf{J}$ is required so that

$$I_t[J_i, J_j]I_t^{-1} = I_t i\hbar\epsilon_{ijk}J_k I_t^{-1}$$

Scattering amplitude

Suppose we have a state in a basis of kets $\mid n\rangle$ and its time reversed counterparts $\mid n^R\rangle$ and $I_t\mid n\rangle = \mid n^R\rangle$. Then

$$\begin{aligned}
\mid \psi\rangle &= \sum_n \mid n\rangle\langle n \mid \psi\rangle \\
I_t\mid \psi\rangle &= \mid \psi^R\rangle = \sum_n \mid n^R\rangle\langle n \mid \psi\rangle^* = \sum_n U\mid n\rangle\langle n \mid \psi\rangle^*
\end{aligned}$$

Also

$$\begin{aligned}
\mid \phi\rangle &= \sum_m \mid m\rangle\langle n \mid \phi\rangle \\
I_t\mid \phi\rangle &= = \mid \phi^R\rangle = \sum_n U\mid m\rangle\langle m \mid \phi\rangle^*
\end{aligned}$$

Then

$$\left\langle \phi^R \mid \psi^R\right\rangle = \sum_{m,n}\langle m \mid \phi\rangle\left\langle m \mid U^\dagger U \mid n\right\rangle\langle n \mid \psi\rangle^* = \langle \psi \mid \phi\rangle$$

So suppose we have $\mid \psi\rangle$ as the initial state and $\mid \phi\rangle$ as the final state and we are interested in the amplitude

$$\langle \phi \mid A \mid \psi\rangle$$

where A is any operator. We showed that the forward and reversed amplitudes are related according to

$$\left\langle \psi^R \mid I_t A I_t^{-1} \mid \phi^R \right\rangle = \langle \phi \mid A \mid \psi \rangle$$

If A is Hermitian and invariant under time reversal

$$\left\langle \psi^R \mid A \mid \phi^R \right\rangle = \langle \phi \mid A \mid \psi \rangle$$

Wave function

How does the wave function transform? First let's note that if $\psi(t)$ is a solution to Schrodinger's equation then $\psi(-t)$ in general is not because of that first derivative. On the other hand $\psi^*(-t)$ is a solution as we can see by taking the complex conjugate of

$$i\hbar \frac{\partial}{\partial t}\psi(t) = H\psi(t)$$

where H is real. More generally,

$$\langle \mathbf{x} \mid \alpha, t \rangle = \left\langle \mathbf{x} \mid e^{-iHt/\hbar} \mid \alpha \right\rangle, \quad \left\langle \mathbf{x} \mid \alpha, t^R \right\rangle = \left\langle \mathbf{x} \mid e^{-iHt/\hbar} \mid \alpha^R \right\rangle$$

Now $\mid \mathbf{x}\rangle$ is invariant under time reversal so

$$\left\langle \mathbf{x} \mid e^{-iHt/\hbar} \mid \alpha^R \right\rangle = \left\langle \mathbf{x}^R \mid e^{-iHt/\hbar} \mid \alpha^R \right\rangle = \left\langle \alpha \mid I_t e^{iHt/\hbar} I_t^{-1} \mid \mathbf{x} \right\rangle = \left\langle \alpha \mid e^{-iHt/\hbar} \mid \mathbf{x} \right\rangle$$

$$\rightarrow \psi_R(x,t) = \psi^*(x,-t)$$

The angular part of the wave function written in terms of spherical harmonics will transform according to

$$Y_{lm} \rightarrow Y_{lm}^* = (-1)^m Y_{l,-m}$$

which suggests that

$$I_t \mid l, m\rangle = (-1)^m \mid l, -m\rangle$$

and

$$I_t^2 \mid l, m\rangle = \mid l, m\rangle \rightarrow I_t^2 = 1$$

Spin 1/2

We have established that $I_t \mathbf{J} I_t^{-1} = -\mathbf{J}$ so it stands to reason that for spin 1/2 that

$$I_t \sigma I_t^{-1} = UK\sigma KU^\dagger = U\sigma^* U^\dagger = -\sigma$$

which means that

$$U\sigma_x U^\dagger = -\sigma_x, \quad U\sigma_y U^\dagger = \sigma_y, \quad U\sigma_z U^\dagger = -\sigma_z$$

where we have used

$$\begin{aligned} \sigma_i\sigma_j + \sigma_j\sigma_i &= 2\delta_{ij} \\ \sigma_i\sigma_j &= 2\delta_{ij} - \sigma_j\sigma_i \\ \rightarrow \sigma_i\sigma_j\sigma_i^{-1} &= 2\delta_{ij}\sigma_i^{-1} - \sigma_j \\ \rightarrow U &= e^{i\delta}\sigma_y \end{aligned}$$

Then

$$I_t|+\rangle = e^{i\delta}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = ie^{i\delta}\begin{pmatrix}0\\1\end{pmatrix} = ie^{i\delta}|-\rangle$$

$$I_t|-\rangle = e^{i\delta}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = -ie^{i\delta}\begin{pmatrix}1\\0\end{pmatrix} = -ie^{i\delta}|+\rangle$$

Also $I_t^2|+\rangle = e^{i\delta}\sigma_y K\left(ie^{i\delta}|-\rangle\right) = e^{i\delta}(-ie^{-i\delta})(-i)|+\rangle = -1$ In general

$$I_t^2 = e^{i\delta}\sigma_y K e^{i\delta}\sigma_y K = -\sigma_y^2 = -1$$

For integer spin $I_t^2 = 1$ and for half integer spin $I_t^2 = -1$. For half integer spin, $I_t^2 = (-1)^N$ where N is the number of half integer spin particles.

We can generalize to arbitrary spin by noting that an eigenket in the direction $\hat{\mathbf{n}}$ can be written

$$\begin{aligned} |\,\hat{\mathbf{n}},+\rangle &= e^{-iJ_z\alpha/\hbar}e^{-iJ_y\beta/\hbar}|+\rangle \\ I_t|\,\hat{\mathbf{n}},+\rangle &= e^{-iJ_z\alpha/\hbar}e^{-iJ_y\beta/\hbar}I_t|+\rangle = \eta|\,\hat{\mathbf{n}},-\rangle \end{aligned}$$

Meanwhile we might have written

$$|\,\hat{\mathbf{n}},-\rangle = e^{-iJ_z\alpha/\hbar}e^{-iJ_y(\pi+\beta)/\hbar}|+\rangle$$

Together we get

$$I_t = \eta e^{-iJ_y\pi/\hbar}K$$

It reduces to $I_t = e^{i\delta}\frac{2S_y}{\hbar}K$ when $j = \frac{1}{2}$. Now

$$I_t^2 = \eta e^{-iJ_y\pi/\hbar}K\eta e^{-iJ_y\pi/\hbar}K = \eta e^{-iJ_y\pi/\hbar}\eta^* e^{iJ_y^*\pi/\hbar} = |\eta|^2 e^{-i2\pi J_y/\hbar}$$

where we have used $J_y^* = -J_y$. It is evident that for half integer j, $I_t^2 = -1$.

1.2 Symmetry Properties of Scattering Amplitudes

If the interaction is rotationally invariant, the amplitude can depend only on $|\mathbf{k}|^2, |\mathbf{k}'|^2$ and $\mathbf{k}\cdot\mathbf{k}'$. Then the scattering amplitude depends only on k and the scattering angle θ.

$$f(\mathbf{k}',\mathbf{k}) = f(k,\theta)$$

If the interaction is reflection invariant

$$f(\mathbf{k}',\mathbf{k}) = f(-\mathbf{k}',-\mathbf{k})$$

What about time reversal. For any two states $\mid a\rangle$ and $\mid b\rangle$, operator A, and time reversal operator I_t

$$\langle a \mid A \mid b\rangle = \left\langle b^R \mid I_t A^\dagger I_t^{-1} \mid a^R\right\rangle.$$

If the scattering potential V is invariant with respect to time reversal then

$$\begin{aligned} \mid \mathbf{k}^+\rangle &= \mid \mathbf{k}\rangle + \frac{1}{E-H_0+i\epsilon}V\mid \mathbf{k}^+\rangle \\ I_t\mid \mathbf{k}^+\rangle &= \mid -\mathbf{k}\rangle + \frac{1}{E-H_0-i\epsilon}VI_t\mid \mathbf{k}^+\rangle \end{aligned}$$

Define

$$\mid \mathbf{k}^-\rangle = \mid \mathbf{k}\rangle + \frac{1}{E-H_0-i\epsilon}V\mid \mathbf{k}^-\rangle$$

as the state with incoming spherical wave that scatters into the momentum eigenstate $\mathbf{k}$. Then

$$I_t\mid \mathbf{k}^+\rangle = \mid -\mathbf{k}^-\rangle$$

Under time reversal incoming and outgoing states are reversed.

Another way to write an expression for the scattered state is to recognize that

$$\begin{aligned} (E-H_0)\mid \mathbf{k}\rangle &= 0 \\ \rightarrow (E-H)\mid \mathbf{k}\rangle &= -V\mid \mathbf{k}\rangle \\ \rightarrow \mid \mathbf{k}\rangle &= \mid \mathbf{k}^\pm\rangle - \frac{1}{E-H\pm i\epsilon}V\mid \mathbf{k}\rangle \\ \rightarrow \mid \mathbf{k}^\pm\rangle &= \left(1+\frac{1}{E-H\pm i\epsilon}V\right)\mid \mathbf{k}\rangle \end{aligned}$$

Next define:

$$T = V + V\frac{1}{E-H\pm \imath\epsilon}V$$

Then

$$\begin{aligned} \left\langle \mathbf{p} \mid V \mid \mathbf{k}^+\right\rangle &= \langle \mathbf{p} \mid T(E) \mid \mathbf{k}\rangle \quad \text{iff } E_k = E_p = E, \\ &= \left\langle \mathbf{p}^- \mid V \mid \mathbf{k}\right\rangle \quad \text{iff } E_p = E_k. \end{aligned}$$

Now the effect of time reversal is

$$\langle \mathbf{p} \mid T \mid \mathbf{k} \rangle = \left\langle -\mathbf{k} \mid I_t T^\dagger I_t^{-1} \mid -\mathbf{p} \right\rangle$$

because $\mid \mathbf{k}^R \rangle = \mid -\mathbf{k} \rangle$. If V is invariant under time reversal,

$$I_t T^\dagger I_t^{-1} = T$$

and

$$\left\langle -\mathbf{k} \mid I_t T^\dagger I_t^{-1} \mid -\mathbf{p} \right\rangle = \langle -\mathbf{k} \mid T \mid -\mathbf{p} \rangle$$

and

$$f(\mathbf{p}, \mathbf{k}) = f(-\mathbf{k}, -\mathbf{p}).$$