1.1.1 Time reversal

Time reversal is a tricky business. First consider a classical example. Suppose a planet is in orbit about the sun. At some time say \( t = 0 \) it has a velocity \( \mathbf{v}(0) \). If we reverse the velocity so that \( \mathbf{v}(0) \rightarrow -\mathbf{v}(0) \) then the planet will retrace it’s trajectory so that \( x_R(t) = x(-t) \) and \( \mathbf{v}_R(t) = -\mathbf{v}(-t) \) where \( x_R(t) \) is the position of the reversed planet at \( t \) and \( \mathbf{v}_R(t) \) is the velocity of the reversed planet at time \( t \). So we reverse the state and propagate by \( t \) and we should end up with exactly the same position and velocity as if we had propagated the original state by \( -t \) and then reversed the velocity.

Now quantum mechanics. We have a state \( |\psi\rangle \) and \( |\psi_R\rangle \) the reversed state and suppose we have some operator that effects the reversal so that

\[
I_t |\psi\rangle = |\psi_R\rangle
\]

If \( |\psi\rangle = |p'\rangle \) represents a state with definite momentum then \( |\psi_R\rangle = |-p'\rangle \) and we expect that

\[
\mathbf{p} |\psi_R\rangle = -\mathbf{p}' |\psi_R\rangle
\]

and

\[
\langle \psi_R | \mathbf{p} |\psi_R\rangle = -\langle \psi | \mathbf{p} |\psi\rangle \rightarrow I_t^{-1} \mathbf{p} I_t = -\mathbf{p}
\]

So far so good. The reversed state at time \( t \), \( |\psi(t)^R\rangle \), should be the same as the original state at \( -t \) and then reversed. So we have the state at \( t = 0 \) and we reverse it and propagate it by an infinitesimal time \( \delta t \). Meanwhile we propagate the original state by \( -\delta t \) and then reverse it and we should end up in the same place. That is

\[
(1 - \frac{i}{\hbar} H \delta t) I_t |\psi\rangle = I_t (1 + \frac{i}{\hbar} H \delta t) |\psi\rangle \rightarrow i H I_t = -I_t i H
\]

1.1 Unitary and anti-unitary operators

The parity operator \( \pi \), like all of the others that we have discussed, is unitary. We found that \( \pi^2 |\alpha\rangle = |\alpha\rangle \) so we see that \( \pi \pi = 1 \). Therefore \( \pi = \pi^{-1} \) and \( \pi = \pi^\dagger \). Finally \( \langle \alpha | \pi^{-1} \pi | \beta \rangle = \langle \alpha | \beta \rangle \). The parity operator preserves the inner product.

We like unitary operators so that the inner product is invariant. In particular, if \( U \) is unitary then \( U|\alpha\rangle = |\alpha'\rangle \) and \( U|\beta\rangle = |\beta'\rangle \) and

\[
\langle \alpha' | \beta' \rangle = \langle \alpha | U^\dagger U | \beta \rangle = \langle \alpha | \beta \rangle
\]

which is nice but perhaps not essential. What is essential is that

\[
|\langle \alpha' | \beta' \rangle| = |\langle \alpha | \beta \rangle|.
\]

It turns out that this transformation is necessarily unitary or anti-unitary. Those are the only possibilities. As we will see in a moment a unitary operator does not work for time reversal. We consider an anti-unitary operator \( I_t \) such that \( I_t |\alpha\rangle = |\alpha'\rangle \) and \( I_t |\beta\rangle = |\beta'\rangle \) but now

\[
\langle \alpha' | \beta' \rangle = \langle \alpha | \beta \rangle^* \]
so we preserve length but perhaps not phase. In practice if

$$| \psi \rangle = a_1 | 1 \rangle + a_2 | 2 \rangle$$

then

$$I_t | \psi \rangle = a_1^* I_t | 1 \rangle + a_2^* I_t | 2 \rangle$$

In general we can write an anti-unitary operator as the product of a unitary operator and the complex conjugation operator. That is

$$I_t = U K, \quad I_t^{-1} = K U^\dagger$$

where $U$ is unitary and $K$ takes the complex conjugate of the coefficients. The anti-unitary transformation is not useful as a member of a continuous group because applying $I_t$ twice results in a unitary transformation and we like the composition rule where we can apply two transformations to get a third. The only candidates for an anti-unitary transformation are where $T^2$ gets you back where you started, namely parity and time reversal, charge conjugation, interchange.

**Transformation of $H$ under time reversal**

Now back to the problem at hand. Referring back to Equation 1.1, if $I_t$ is unitary then we conclude that $H I_t = -I_t H$ and for some eigenket of $H$

$$I_t H | n \rangle = I_t E_n | n \rangle = E_n I_t | n \rangle = -I_t H | n \rangle = -E_n I_t | n \rangle$$

which says that we have an eigenket of $H$ with negative energy. That makes no sense. If we have a free particle and turn it around, we don’t get negative energy since energy scales as $p^2$. The other choice is that $I_t$ is anti-unitary. Then Equation 1.1 gives us

$$H I_t = I_t H \rightarrow [H, I_t] = 0$$

**Anti-unitary transformation of expectation value**

Suppose that

$$| \bar{\alpha} \rangle = I_t | \alpha \rangle, \quad | \bar{\beta} \rangle = I_t | \beta \rangle$$

and $I_t$ is anti-unitary. Then

$$\langle \bar{\beta} | \bar{\alpha} \rangle = \langle \alpha | \beta \rangle$$

and if $| \gamma \rangle = A^\dagger | \beta \rangle$ then

$$\langle \beta | A | \alpha \rangle = \langle \gamma | \alpha \rangle = \langle \bar{\alpha} | I_t A^\dagger | \beta \rangle = \langle \bar{\alpha} | I_t A^\dagger I_t^{-1} I_t | \beta \rangle = \langle \bar{\alpha} | I_t A I_t^{-1} | \bar{\beta} \rangle$$

So the rule for the anti-unitary operator $I_t$ is that for any linear operator $A$,

$$\langle \beta | A | \alpha \rangle = \langle \bar{\alpha} | I_t A I_t^{-1} | \bar{\beta} \rangle$$

If $A$ is Hermitian then

$$\langle \beta | A | \alpha \rangle = \langle \bar{\alpha} | I_t A I_t^{-1} | \bar{\beta} \rangle$$

An observable is even or odd if

$$I_t A I_t^{-1} = \pm A.$$

Therefore

$$\langle \alpha | A | \alpha \rangle = \pm \langle \bar{\alpha} | I_t A I_t^{-1} | \bar{\alpha} \rangle$$
Transformation of position and angular momentum

The expectation value of position does not change under time reversal so

$$I_t^{-1}xI_t = x$$

In order that time reversal operation to effect no geometrical transformation it must be that angular momentum is odd. Then

$$I_t^{-1}D I_t = I_t^{-1}e^{-\frac{i}{\hbar} J \cdot \phi} I_t = e^{\frac{-i}{\hbar} J \cdot \phi} I_t = D$$

as long as $I_t^{-1}J I_t = -J$ and this is also in agreement with the notion that $x$ is even and $p$ is odd.

Uncertainty principle

In view of the above

$$[x_i, p_j] = i\hbar \delta_{ij}$$

$$I_t[x_i, p_j] I_t^{-1} = I_t i\hbar \delta_{ij} I_t^{-1}$$

$$[x_i, -p_j] = -i\hbar \delta_{ij}$$

so the uncertainty principle hangs together. Note that $I_t J I_t^{-1} = -J$ is required so that

$$I_t [J_i, J_j] I_t^{-1} = i\hbar \epsilon_{ijk} J_k I_t^{-1}$$

Scattering amplitude

Suppose we have a state in a basis of kets $| n \rangle$ and its time reversed counterparts $| n^R \rangle$ and $I_t | n \rangle = | n^R \rangle$. Then

$$| \psi \rangle = \sum_n | n \rangle \langle n | \psi \rangle$$

$$I_t | \psi \rangle = | \psi^R \rangle = \sum_n | n^R \rangle \langle n | \psi \rangle^* = \sum_n U^| n \rangle \langle n | \psi \rangle^*$$

Also

$$| \phi \rangle = \sum_m | m \rangle \langle n | \phi \rangle$$

$$I_t | \phi \rangle = | \phi^R \rangle = \sum_n U^| m \rangle \langle m | \phi \rangle^*$$

Then

$$\langle \phi^R | \psi^R \rangle = \sum_{m,n} \langle m | \phi \rangle \langle m | U^U | n \rangle \langle n | \psi \rangle^* = \langle \psi | \phi \rangle$$

So suppose we have $| \psi \rangle$ as the initial state and $| \phi \rangle$ as the final state and we are interested in the amplitude

$$\langle \phi | A | \psi \rangle$$
where $A$ is any operator. We showed that the forward and reversed amplitudes are related according to

$$\langle \psi^R | I_t A I_t^{-1} | \phi^R \rangle = \langle \phi | A | \psi \rangle$$

If $A$ is Hermitian and invariant under time reversal

$$\langle \psi^R | A | \phi^R \rangle = \langle \phi | A | \psi \rangle$$

**Wave function**

How does the wave function transform? First let’s note that if $\psi(t)$ is a solution to Schrodinger’s equation then $\psi(-t)$ in general is not because of that first derivative. On the other hand $\psi^*(-t)$ is a solution as we can see by taking the complex conjugate of

$$i\hbar \frac{\partial}{\partial t} \psi(t) = H \psi(t)$$

where $H$ is real. More generally,

$$\langle x | \alpha, t \rangle \equiv \langle x | e^{-iHt/\hbar} | \alpha \rangle, \quad \langle x | \alpha, t^R \rangle = \langle x | e^{-iHt/\hbar} | \alpha^R \rangle$$

Now $|x\rangle$ is invariant under time reversal so

$$\langle x | e^{-iHt/\hbar} | \alpha^R \rangle = \langle x^R | e^{-iHt/\hbar} | \alpha^R \rangle = \langle \alpha | I_t e^{iHt/\hbar} I_t^{-1} | x \rangle = \langle \alpha | e^{-iHt/\hbar} | x \rangle$$

$$\rightarrow \psi^R(x, t) = \psi^*(x, -t)$$

The angular part of the wave function written in terms of spherical harmonics will transform according to

$$Y_{lm} \rightarrow Y_{lm}^* = (-1)^m Y_{l,-m}$$

which suggests that

$$I_l | l, m \rangle = (-1)^m | l, -m \rangle$$

and

$$I_l^2 | l, m \rangle = | l, m \rangle \rightarrow I_l^2 = 1$$

**Spin 1/2**

We have established that $I_t J I_t^{-1} = -J$ so it stands to reason that for spin 1/2 that

$$I_l \sigma I_l^{-1} = U K \sigma K U^\dagger = U \sigma^* U^\dagger = -\sigma$$

which means that

$$U \sigma_x U^\dagger = -\sigma_x, \quad U \sigma_y U^\dagger = \sigma_y, \quad U \sigma_z U^\dagger = -\sigma_z$$

where we have used

$$\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij}$$

$$\sigma_i \sigma_j = 2\delta_{ij} - \sigma_j \sigma_i$$

$$\rightarrow \sigma_i \sigma_j \sigma_i^{-1} = 2\delta_{ij} \sigma_i^{-1} - \sigma_j$$

$$\rightarrow U = e^{i\sigma_y}$$
Then
\[ I_t |+\rangle = e^{i\delta} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = ie^{i\delta} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = ie^{i\delta} |\rangle \\
I_t |\rangle = e^{i\delta} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -ie^{i\delta} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -ie^{i\delta} |+\rangle \]

Also \[ I_t^2 |+\rangle = e^{i\delta} \sigma_y K (ie^{i\delta} |\rangle) = e^{i\delta} (-ie^{-i\delta})(-i) |+\rangle = 1 \]
In general
\[ I_t^2 = e^{i\delta} \sigma_y K e^{i\delta} \sigma_y K = -\sigma_y^2 = -1 \]

For integer spin \( I_t^2 = 1 \) and for half integer spin \( I_t^2 = -1 \). For half integer spin, \( I_t^2 = (-1)^N \) where \( N \) is the number of half integer spin particles.

We can generalize to arbitrary spin by noting that an eigenket in the direction \( \hat{n} \) can be written
\[
| \hat{n}, +\rangle = e^{-iJ_z \alpha/\hbar} e^{-iJ_y \beta/\hbar} |+\rangle \\
I_t | \hat{n}, +\rangle = e^{-iJ_z \alpha/\hbar} e^{-iJ_y \beta/\hbar} I_t |+\rangle = \eta | \hat{n}, -\rangle
\]

Meanwhile we might have written
\[
| \hat{n}, -\rangle = e^{-iJ_z \alpha/\hbar} e^{-iJ_y (\pi+\beta)/\hbar} |+\rangle
\]

Together we get
\[ I_t = \eta e^{-iJ_y \pi/\hbar} K \]

It reduces to \( I_t = e^{i\delta} 2S_z K \) when \( j = \frac{1}{2} \). Now
\[ I_t^2 = \eta e^{-iJ_y \pi/\hbar} K \eta e^{-iJ_y \pi/\hbar} K = \eta e^{-iJ_y \pi/\hbar} \eta^* e^{-iJ_y \pi/\hbar} = |\eta|^2 e^{-i2\pi J_y/\hbar} \]
where we have used \( J_y^* = -J_y \). It is evident that for half integer \( j \), \( I_t^2 = -1 \).
1.2 Symmetry Properties of Scattering Amplitudes

If the interaction is rotationally invariant, the amplitude can depend only on $|k|^2, |k'|^2$ and $k \cdot k'$. Then the scattering amplitude depends only on $k$ and the scattering angle $\theta$.

$$f(k', k) = f(k, \theta)$$

If the interaction is reflection invariant

$$f(k', k) = f(-k', -k)$$

What about time reversal. For any two states $|a\rangle$ and $|b\rangle$, operator $A$, and time reversal operator $I_t$

$$\langle a | A | b \rangle = \langle b^R | I_t A^\dagger I_t^{-1} | a^R \rangle.$$ 

If the scattering potential $V$ is invariant with respect to time reversal then

$$|k^+\rangle = |k\rangle + \frac{1}{E - H_0 + i\epsilon} V |k^+\rangle$$

$$I_t |k^+\rangle = |-k\rangle + \frac{1}{E - H_0 - i\epsilon} V I_t |k^+\rangle$$

Define

$$|k^-\rangle = |k\rangle + \frac{1}{E - H_0 - i\epsilon} V |k^-\rangle$$

as the state with incoming spherical wave that scatters into the momentum eigenstate $k$. Then

$$I_t |k^\pm\rangle = | -k^\mp\rangle$$

Under time reversal incoming and outgoing states are reversed.

Another way to write an expression for the scattered state is to recognize that

$$\langle E - H_0 | k \rangle = 0$$

$$\rightarrow \langle E - H | k \rangle = -V |k\rangle$$

$$\rightarrow |k\rangle = |k^{\pm}\rangle - \frac{1}{E - H \pm i\epsilon} V |k\rangle$$

$$\rightarrow |k^{\pm}\rangle = \left(1 + \frac{1}{E - H \pm i\epsilon} V \right) |k\rangle$$

Next define:

$$T = V + \frac{1}{E - H \pm i\epsilon} V$$

Then

$$\langle p | V | k^+ \rangle = \langle p | T(E) | k \rangle \quad \text{iff} \quad E_k = E_p = E,$$

$$= \langle p^- | V | k \rangle \quad \text{iff} \quad E_p = E_k.$$
Now the effect of time reversal is
\[
\langle p \mid T \mid k \rangle = \langle -k \mid I_t T^\dagger I_{-t}^{-1} \mid -p \rangle
\]
because \( |k^R\rangle = | -k\rangle \). If \( V \) is invariant under time reversal,
\[
I_t T^\dagger I_{-t}^{-1} = T
\]
and
\[
\langle -k \mid I_t T^\dagger I_{-t}^{-1} \mid -p \rangle = \langle -k \mid T \mid -p \rangle
\]
and
\[
f(p, k) = f(-k, -p).
\]