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Lecture XV

1.2 Spin dependent scattering

We begin by generalizing the transition operator to include spin. Then

$$\left\langle \mathbf{k}' \mid T \mid \mathbf{k} \right\rangle \rightarrow \langle \mathbf{k}', \nu_f \mid T \mid \mathbf{k}, \nu_i \rangle \tag{1.1}$$

The scattered state is

$$\psi^+ = \frac{1}{(2\pi)^{3/2}} \left(e^{i(\mathbf{k_i}\cdot\mathbf{r})} | \nu_i\rangle + \frac{e^{ikr}}{r} \sum_{\nu_f} | \nu_f\rangle f(\mathbf{k}_f \nu_f; \mathbf{k}_i \nu_i) \right),$$

so that

$$\frac{d\sigma_{i\rightarrow f}}{d\Omega} = |f(\mathbf{k}_f \nu_f | \mathbf{k}_i \nu_i)|^2.$$

The transition operator now acts in spin space as well as momentum space and we can write

$$\langle \mathbf{k}', \nu_f \mid T \mid \mathbf{k}, \nu_i \rangle \rightarrow \langle \nu_f \mid M(\mathbf{k}', \mathbf{k}) \mid \nu_i \rangle \tag{1.2}$$

Let's construct an operator M that is invariant with respect to parity and time reversal, and rotations. According to that last requirement it is a scalar. If we suppose that initial and final states have spin $\frac{1}{2}$ and the scattering potential is spherically symmetric, (to keep it simple), then it is a 2 x 2 Hermitian operator. The most general such operator is constructed as a linear combination of the identity matrix and the pauli matrices. The available kinematic parameters are the initial and final momenta and spin. The scalar combinations are

$$(\mathbf{k}' \times \mathbf{k}) \cdot \sigma, \quad (\mathbf{k}' - \mathbf{k}) \cdot \sigma, \quad (\mathbf{k}' + \mathbf{k}) \cdot \sigma$$

The first will be a scalar and the second and third pseudo-scalar, (since spin is a pseudo vector).

The most general transition scattering operator has the form

$$M(\mathbf{k}', \mathbf{k}) = g_1 + g_2(\mathbf{k}' \times \mathbf{k}) \cdot \sigma + g_3(\mathbf{k}' - \mathbf{k}) \cdot \sigma + g_4(\mathbf{k}' + \mathbf{k}) \cdot \sigma \tag{1.3}$$

The third and fourth terms are not invariant with respect to parity as the momentum vector changes sign but the angular momentum vector does not. Only the first and second terms are allowed for a parity conserving process. Note also that the third term is not time reversal invariant. Time reversal changes the sign of both momentum and spin and $k_i \rightarrow k_f$. So the most general form for the scattering operator that is rotationally invariant and preserves time and space reversal symmetry is

$$M(\mathbf{k}_f, \mathbf{k}_i) = g_1(k, \theta) + g_2(k, \theta)\sigma \cdot \frac{\mathbf{k}' \times \mathbf{k}}{|\mathbf{k}' \times \mathbf{k}|} = g_1(k, \theta) + g_2(k, \theta)\sigma \cdot \hat{\mathbf{n}} \tag{1.4}$$

where $\hat{\mathbf{n}}$ is the unit vector perpendicular to the scattering plane. Of course Equation **??** is the most general form of the scattering operator. The scattering potential might not interact with the particle spin and then we would have that $g_2 = 0$. But the potential is presumed rotationally invariant, and invariant under reflection so that if g_2 is not zero, then the most general form for the potential is

$$V = V_0(r) + \sigma \cdot \mathbf{L} V_1(r) \tag{1.5}$$

$\mathbf{L}$ being the only available axial vector in the space of the spatial coordinate (and parallel to $\hat{\mathbf{n}}$).

1.3 Spin Dependent Scattering - II

1.3.1 Spin density matrix

If the initial spin state is $\mid \nu_n\rangle$ with probability $p_{i,n}$, then the probability to scatter to final state $\langle \nu_m \mid$ is

$$\begin{aligned}
p_{f,m} &= \sum_n p_{i,n} |\langle \nu_f \mid M \mid \nu_{i,n}\rangle|^2 \\
&= \sum_n p_{i,n} \langle \nu_f \mid M \mid \nu_{i,n}\rangle \left\langle \nu_{i,n} \mid M^\dagger \mid \nu_f \right\rangle \\
&= \langle \nu_m \mid \left[\sum_n M \mid \nu_{i,n}\rangle p_{i,n} \langle \nu_{i,n} \mid M^\dagger \right] \mid \nu_m\rangle \\
\rightarrow \rho_f &= \frac{M\rho_i M^\dagger}{\mathrm{Tr}\rho_i M^\dagger M}
\end{aligned}$$

where ρ_i is the spin density matrix of the initial state

$$\rho_i = \sum_n \mid \nu_n\rangle p_{i,n} \langle \nu_n \mid \qquad (1.6)$$

and ρ_f is the density matrix of the final state. The density matrix is normalized so that $\mathrm{Tr}\rho = 1$. The differential cross section to scatter from $\mid \nu_n\rangle$ with $\mathbf{k}$ to $\mid \nu_m\rangle$ with $\mathbf{k}'$ is

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \sum_{n,m} |\langle \nu_m \mid M(\mathbf{k},\mathbf{k}') \mid \nu_n\rangle|^2 p_{i,n} \\
&= \mathrm{Tr}\rho_i M^\dagger M
\end{aligned}$$

If the initial state is unpolarized, then $\rho_i = \frac{1}{N_s} = \frac{1}{2}$. The differential cross section is

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{1}{2}\mathrm{Tr}M^\dagger M \\
&= \frac{1}{2}\mathrm{Tr}(g^* + \sigma\cdot\hat{\mathbf{n}}h^*)(g + \sigma\cdot\hat{\mathbf{n}}h) \\
&= (|g(k,\theta|^2 + |h(k,\theta|^2) \qquad (1.7)
\end{aligned}$$

The polarization is defined as the net spin. The final state polarization is

$$\begin{aligned}
\mathbf{P_f} &= \mathrm{Tr}\sigma\rho_f \\
&= \frac{\mathrm{Tr}\sigma M^\dagger \rho_i M}{\mathrm{Tr}\rho_i M^\dagger M} \\
&= \frac{\mathrm{Tr}\sigma(g^* + \sigma\cdot\hat{\mathbf{n}}h^*)\rho_i(g + \sigma\cdot\hat{\mathbf{n}}h)}{\mathrm{Tr}\rho_i M^\dagger M} \\
&= \frac{1}{2}\frac{\mathrm{Tr}\sigma(g^* + \sigma\cdot\hat{\mathbf{n}}h^*)(g + \sigma\cdot\hat{\mathbf{n}}h)}{\mathrm{Tr}\rho_i M^\dagger M} \\
&= \frac{\hat{\mathbf{n}}h^* g + g^*\hat{\mathbf{n}}h}{\mathrm{Tr}\rho_i M^\dagger M} \\
&= 2\frac{\hat{\mathbf{n}}\Re(h^* g)}{|g|^2 + |h|^2}
\end{aligned}$$

That there is a final state polarization for scattering from a potential that is spherically symmetric and is in some sense a consequence of parity reversal invariance. Consider a particle traveling from $-\infty$ along the y-axis toward the origin and scattering in the x-y plane (plane of the paper), at an angle θ with respect to y. $\hat{\mathbf{n}}$ is parallel to the z-axis (perpendicular to the plane of the paper). Suppose the scattered particle is polarized so that $\mathbf{P} = P_x\hat{\mathbf{x}} + P_y\hat{\mathbf{y}} + P_z\hat{\mathbf{z}}$.

1. Reflect by imagining a mirror in the x-z plane, (the plane perpendicular to the y-direction). Then $\mathbf{k} \rightarrow -\mathbf{k}$ and $k'_y\hat{\mathbf{j}} \rightarrow -k'_y\hat{\mathbf{j}}$. There is no change to $k'_x\hat{\mathbf{i}}$ The polarization becomes $\mathbf{P} \rightarrow -P_x\hat{\mathbf{x}} + P_y\hat{\mathbf{y}} - P_z\hat{\mathbf{z}}$..

2. Rotate about the x-axis by π. Now $\mathbf{k} \rightarrow -\mathbf{k}$ (back where we started) and $k'_y\hat{\mathbf{j}} \rightarrow -k'_y\hat{\mathbf{j}}$, also back where we started. And $\mathbf{P} \rightarrow -P_x\hat{\mathbf{x}} - P_y\hat{\mathbf{y}} + P_z\hat{\mathbf{z}}$.

If there were a component of polarization in the x or y direction, the observer in the inverted world would would find that the scattered particle ended up with a different polarization. If parity is a good symmetry then the scattering amplitude must be the same in the inverted world. Only polarization in the z-direction would remain unchanged after the transformations described so that is all that is allowed.

1.3.2 Polarization measurement

Now suppose that the initial state has some net polarization $\mathbf{P_i}$. The density matrix

$$\rho_i = \frac{1}{2}(1 + \sigma \cdot \mathbf{P_i}) \tag{1.8}$$

so that $\mathbf{P_i} = \mathrm{Tr}\sigma\rho_i = \mathbf{P_i}$. Then

$$\begin{aligned}
\mathbf{P_f} &= \frac{\mathrm{Tr}\sigma M^\dagger \rho_i M}{\mathrm{Tr}(\rho_i M^\dagger M)} \\
&= \frac{\mathrm{Tr}\sigma(g^* + \sigma\cdot\hat{\mathbf{n}}h^*)\rho_i(g + \sigma\cdot\hat{\mathbf{n}}h)}{\mathrm{Tr}(\rho_i M^\dagger M)} \\
&= \frac{\mathrm{Tr}\sigma(g^* + \sigma\cdot\hat{\mathbf{n}}h^*)\frac{1}{2}(1 + \sigma\cdot\mathbf{P_i})(g + \sigma\cdot\hat{\mathbf{n}}h)}{\mathrm{Tr}(\rho_i M^\dagger M)} \\
&= \frac{2\hat{\mathbf{n}}\Re(h^* g) + (|g|^2 + |h|^2)\mathbf{P_i}}{|g|^2 + |h|^2 + 2\Re(g^* h)\mathbf{P_i}\cdot\hat{\mathbf{n}}}
\end{aligned}$$

What we want to know is the differential cross section (Equation 1.2).

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \mathrm{Tr}\rho_i M^\dagger M \\
&= \frac{1}{2}\mathrm{Tr}(1 + \hat{\mathbf{n}}\cdot P_i)(g^* + \sigma\cdot\hat{\mathbf{n}}h^*)(g + \sigma\cdot\hat{\mathbf{n}}h) \\
&= (|g(k,\theta)|^2 + |h(k,\theta)|^2 + 2\Re(g^* h)\mathbf{P_i}\cdot\hat{\mathbf{n}})
\end{aligned}$$

We find that the cross section depends on the direction of the polarization with respect to the normal to the scattering plane and we want to exploit that dependence to measure the polarization $\mathbf{P}_i$ and the relative size of g, the spin independent amplitude and h the spin dependent amplitude.

Suppose that the beam is traveling in the y-direction $\mathbf{P_i}$ is in the z- direction and it scatters in the x-y plane. Place the left detector at angle θ to the left of the y-axis, and the right detector at angle θ to the right of the y-axis. The normal $\hat{\mathbf{n}}$ will be in the positive z-direction for particles that scatter into the left detector and in the negative z-direction for particles that scatter into the right detector. Then the rate into the left and right detectors will depend on the polarization.

$$\begin{aligned}\frac{d\sigma}{d\Omega}(L) &= (|g(k,\theta)|^2 + |h(k,\theta)|^2 + 2\Re(g^*h)P_i) \\ \frac{d\sigma}{d\Omega}(R) &= (|g(k,\theta)|^2 + |h(k,\theta)|^2 - 2\Re(g^*h)P_i)\end{aligned}$$

and proportional to the asymmetry

$$A = \frac{\sigma_L - \sigma_R}{\sigma_L + \sigma_R} = \frac{2\Re(g^*h)P_i}{(|g|^2 + |h|^2)}$$

1.3.3 Double scattering

To determine $\Re gh^*$, send in unpolarized projectile from the left. It scatters in the x-y plane so that $\hat{\mathbf{n}}$ is in the z-direction. Then scatter again and also just look at events in the x-y plane. Then measure up down $\pm z$ asymmetry. The polarization after the first scatter is $\mathbf{P_i}$ and the up-down asymmetry after the second A.

$$\begin{aligned}\mathbf{P_i} &= \hat{\mathbf{n}}\frac{2\Re gh^*}{|g|^2 + |h|^2} \\ A &= \frac{2P_i\Re gh^*}{|g|^2 + |h|^2} \\ &= \left(\frac{2\Re gh^*}{|g|^2 + |h|^2}\right)^2\end{aligned}$$