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Lecture XVI

1.2 Spin Dependent Scattering - II

1.2.1 Spin density matrix

If the initial spin state is $\mid \nu_n\rangle$ with probability $p_{i,n}$, then the probability to scatter to final state $\langle \nu_m \mid$ is

$$\begin{aligned} p_{f,m} &= \sum_n p_{i,n} |\langle \nu_f \mid M \mid \nu_{i,n}\rangle|^2 \\ &= \sum_n p_{i,n} \langle \nu_f \mid M \mid \nu_{i,n}\rangle \langle \nu_{i,n} \mid M^\dagger \mid \nu_f\rangle \\ &= \langle \nu_m \mid \left[\sum_n M| \nu_{i,n}\rangle p_{i,n} \langle \nu_{i,n} |M^\dagger\right] \mid \nu_m\rangle \\ \rightarrow \rho_f &= \frac{M\rho_i M^\dagger}{\mathrm{Tr}\rho_i M^\dagger M} \end{aligned}$$

where ρ_i is the spin density matrix of the initial state

$$\rho_i = \sum_n \mid \nu_n\rangle p_{i,n} \langle \nu_n \mid \tag{1.1}$$

and ρ_f is the density matrix of the final state. The density matrix is normalized so that $\mathrm{Tr}\rho = 1$. The differential cross section to scatter from $\mid \nu_n\rangle$ with $\mathbf{k}$ to $\mid \nu_m\rangle$ with $\mathbf{k}'$ is

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \sum_{n,m} |\langle \nu_m \mid M(\mathbf{k},\mathbf{k}') \mid \nu_n\rangle|^2 p_{i,n} \\ &= \mathrm{Tr}\rho_i M^\dagger M \end{aligned}$$

If the initial state is unpolarized, then $\rho_i = \frac{1}{N_s} = \frac{1}{2}$. The differential cross section is

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{1}{2}\mathrm{Tr} M^\dagger M \\ &= \frac{1}{2}\mathrm{Tr}(g^* + \sigma\cdot\hat{\mathbf{n}}h^*)(g + \sigma\cdot\hat{\mathbf{n}}h) \\ &= (|g(k,\theta|^2 + |h(k,\theta|^2) \end{aligned} \tag{1.2}$$

The polarization is defined as the net spin. The final state polarization is

$$\begin{aligned} \mathbf{P_f} &= \mathrm{Tr}\sigma\rho_f \\ &= \frac{\mathrm{Tr}\sigma M^\dagger \rho_i M}{\mathrm{Tr}\rho_i M^\dagger M} \\ &= \frac{\mathrm{Tr}\sigma(g^* + \sigma\cdot\hat{\mathbf{n}}h^*)\rho_i(g + \sigma\cdot\hat{\mathbf{n}}h)}{\mathrm{Tr}\rho_i M^\dagger M} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\frac{\mathrm{Tr}\sigma(g^* + \sigma\cdot\hat{\mathbf{n}}h^*)(g+\sigma\cdot\hat{\mathbf{n}}h)}{\mathrm{Tr}\rho_i M^\dagger M} \\
&= \frac{\hat{\mathbf{n}}h^* g + g^*\hat{\mathbf{n}}h}{\mathrm{Tr}\rho_i M^\dagger M} \\
&= 2\frac{\hat{\mathbf{n}}\Re(h^* g)}{|g|^2+|h|^2}
\end{aligned}
$$

That there is a final state polarization for scattering from a potential that is spherically symmetric and is in some sense a consequence of parity reversal invariance. Consider a particle traveling from $-\infty$ along the y-axis toward the origin and scattering in the x-y plane (plane of the paper), at an angle θ with respect to y. $\hat{\mathbf{n}}$ is parallel to the z-axis (perpendicular to the plane of the paper). Suppose the scattered particle is polarized so that $\mathbf{P} = P_x\hat{\mathbf{x}} + P_y\hat{\mathbf{y}} + P_z\hat{\mathbf{z}}$.

1. Reflect by imagining a mirror in the x-z plane, (the plane perpendicular to the y-direction). Then $\mathbf{k} \to -\mathbf{k}$ and $k'_y\hat{\mathbf{j}} \to -k'_y\hat{\mathbf{j}}$. There is no change to $k'_x\hat{\mathbf{i}}$ The polarization becomes $\mathbf{P} \to -P_x\hat{\mathbf{x}} + P_y\hat{\mathbf{y}} - P_z\hat{\mathbf{z}}$..
2. Rotate about the x-axis by π. Now $\mathbf{k} \to -\mathbf{k}$ (back where we started) and $k'_y\hat{\mathbf{j}} \to -k'_y\hat{\mathbf{j}}$, also back where we started. And $\mathbf{P} \to -P_x\hat{\mathbf{x}} - P_y\hat{\mathbf{y}} + P_z\hat{\mathbf{z}}$.

If there were a component of polarization in the x or y direction, the observer in the inverted world would would find that the scattered particle ended up with a different polarization. If parity is a good symmetry then the scattering amplitude must be the same in the inverted world. Only polarization in the z-direction would remain unchanged after the transformations described so that is all that is allowed.

1.2.2 Polarization measurement

Now suppose that the initial state has some net polarization $\mathbf{P_i}$. The density matrix

$$
\rho_i = \frac{1}{2}(1 + \sigma\cdot\mathbf{P_i}) \tag{1.3}
$$

so that $\mathbf{P_i} = \mathrm{Tr}\sigma\rho_i = \mathbf{P_i}$. Then

$$
\begin{aligned}
\mathbf{P_f} &= \frac{\mathrm{Tr}\sigma M^\dagger \rho_i M}{\mathrm{Tr}(\rho_i M^\dagger M)} \\
&= \frac{\mathrm{Tr}\sigma(g^* + \sigma\cdot\hat{\mathbf{n}}h^*)\rho_i(g+\sigma\cdot\hat{\mathbf{n}}h)}{\mathrm{Tr}(\rho_i M^\dagger M)} \\
&= \frac{\mathrm{Tr}\sigma(g^* + \sigma\cdot\hat{\mathbf{n}}h^*)\frac{1}{2}(1+\sigma\cdot\mathbf{P_i})(g+\sigma\cdot\hat{\mathbf{n}}h)}{\mathrm{Tr}(\rho_i M^\dagger M)} \\
&= \frac{2\hat{\mathbf{n}}\Re(h^* g) + (|g|^2+|h|^2)\mathbf{P_i}}{|g|^2+|h|^2+2\Re(g^* h)\mathbf{P_i}\cdot\hat{\mathbf{n}}}
\end{aligned}
$$

What we want to know is the differential cross section (Equation 1.7).

$$
\frac{d\sigma}{d\Omega} = \mathrm{Tr}\rho_i M^\dagger M
$$

$$
\begin{aligned}
&= \frac{1}{2}\mathrm{Tr}(1+\hat{\mathbf{n}}\cdot P_i)(g^*+\sigma\cdot\hat{\mathbf{n}}h^*)(g+\sigma\cdot\hat{\mathbf{n}}h) \\
&= (|g(k,\theta)|^2+|h(k,\theta)|^2+2\Re(g^*h)\mathbf{P_i}\cdot\hat{\mathbf{n}})
\end{aligned}
$$

We find that the cross section depends on the direction of the polarization with respect to the normal to the scattering plane and we want to exploit that dependence to measure the polarization $\mathbf{P}_i$ and the relative size of g, the spin independent amplitude and h the spin dependent amplitude.

Suppose that the beam is traveling in the y-direction $\mathbf{P_i}$ is in the z- direction and it scatters in the x-y plane. Place the left detector at angle θ to the left of the y-axis, and the right detector at angle θ to the right of the y-axis. The normal $\hat{\mathbf{n}}$ will be in the positive z-direction for particles that scatter into the left detector and in the negative z-direction for particles that scatter into the right detector. Then the rate into the left and right detectors will depend on the polarization.

$$
\begin{aligned}
\frac{d\sigma}{d\Omega}(L) &= (|g(k,\theta)|^2+|h(k,\theta)|^2+2\Re(g^*h)P_i) \\
\frac{d\sigma}{d\Omega}(R) &= (|g(k,\theta)|^2+|h(k,\theta)|^2-2\Re(g^*h)P_i)
\end{aligned}
$$

and proportional to the asymmetry

$$
A = \frac{\sigma_L-\sigma_R}{\sigma_L+\sigma_R} = \frac{2\Re(g^*h)P_i}{(|g|^2+|h|^2)}
$$

1.2.3 Double scattering

To determine $\Re gh^*$, send in unpolarized projectile from the left. It scatters in the x-y plane so that $\hat{\mathbf{n}}$ is in the z-direction. Then scatter again and also just look at events in the x-y plane. Then measure up down $\pm z$ asymmetry. The polarization after the first scatter is $\mathbf{P_i}$ and the up-down asymmetry after the second A.

$$
\begin{aligned}
\mathbf{P_i} &= \hat{\mathbf{n}}\frac{2\Re gh^*}{|g|^2+|h|^2} \\
A &= \frac{2P_i\Re gh^*}{|g|^2+|h|^2} \\
&= \left(\frac{2\Re gh^*}{|g|^2+|h|^2}\right)^2
\end{aligned}
$$

Proton Neutron Scattering

Protons and neutrons are both spin 1/2. We will need to extend our scattering matrix to 4 dimensions to include all the possible spin combinations fof the two particles. A proton and neutron exist in a bound state as a deuteron with zero orbital angular momentum and one unit of total angular momentum. $J = 1$. There is no $j = 0$ or $l = 1$ bound state. Therefore the forces between the proton and neutron when the spins are aligned, is evidently different than when the spins are opposite. At low energy, scattering is all s-wave and there will be no mixing of the single and triplet states. We then write the scattering potential as

$$V = V_s + V_t$$

and the scattering matrix becomes

$$M = f_s(\theta)P_s + f_t(\theta)P_t$$

where P_s and P_t are the projection matrices for single and triplet states respectively.

1.2.4 Projection operators

We can determine the projection matrices by noting that the total angular momentum of two spin $\frac{1}{2}$ states is

$$\mathbf{J} = \frac{1}{2}(\sigma_{\mathbf{p}} + \sigma_{\mathbf{n}})$$

and that

$$\mathbf{J^2}|\ \rangle = j(j+1)|\ \rangle$$

For a singlet state, $\mathbf{J^2}|\ s\rangle = 0$ and for a triplet state $\mathbf{J^2}|\ t\rangle = 2$ and for the state that is some mixture of triplet and singlet

$$|\ \psi\rangle = a|\ t\rangle + b|\ s\rangle,$$

the operator $\mathbf{J^2}$ projects out the triplet component,

$$J^2|\ \psi\rangle = 2|\ t\rangle$$

Therefore

$$\begin{aligned} P_t &= N\mathbf{J^2} = N\left(\frac{1}{2}(\sigma_{\mathbf{p}} + \sigma_{\mathbf{n}})\right)^2 \\ &= N\frac{1}{4}({\sigma_{\mathbf{n}}}^2 + {\sigma_{\mathbf{p}}}^2 + 2\sigma_{\mathbf{n}}\cdot\sigma_{\mathbf{p}}) \\ &= N\tfrac{2}{4}(3 + \sigma_{\mathbf{n}}\cdot\sigma_{\mathbf{p}}) \end{aligned}$$

The normalization is determined by requiring

$$\begin{aligned} P_t &= P_t^2 = N^2\frac{1}{4}(9 + 6\sigma_{\mathbf{n}}\cdot\sigma_{\mathbf{p}} + (\sigma_{\mathbf{n}}\cdot\sigma_{\mathbf{p}})^2) \\ &= N^2\frac{1}{4}(9 + 6\sigma_{\mathbf{n}}\cdot\sigma_{\mathbf{p}} + (3 - 2\sigma_{\mathbf{n}}\cdot\sigma_{\mathbf{p}})) \end{aligned}$$

$$\begin{aligned} &= N^2\frac{1}{4}(12+4\sigma_{\mathbf{n}}\cdot\sigma_{\mathbf{p}}) \\ &= N^2(3+\sigma_{\mathbf{n}}\cdot\sigma_{\mathbf{p}}) \\ \rightarrow N &= \frac{1}{4} \end{aligned}$$

so that

$$P_t = \frac{1}{4}(3+\sigma_{\mathbf{p}}\cdot\sigma_{\mathbf{n}})$$

Then since $P_s + P_t = 1$ we find

$$P_s = \frac{1}{4}(1-\sigma_{\mathbf{p}}\cdot\sigma_{\mathbf{n}})$$

1.2.5 Density matrix and differential cross section

The differential cross section for neutron proton scattering is

$$\frac{d\sigma}{d\Omega} = \mathrm{Tr}\rho_i M^\dagger M$$

where ρ_i is the spin density matrix of the initial state. As each of the initial state proton and neutron can be in one of two spin states. Therefore density operator for the initial state is a 4X4 matrix. Remembering that the density operator is a Hermitian matrix with unit trace, there are 15 free parameters. We can write the spin density matrix as

$$\rho = \frac{1}{4}(1+\mathbf{P_p}\cdot\sigma_{\mathbf{p}}+\mathbf{P_n}\cdot\sigma_{\mathbf{n}}+\sum_{ij}C_{ij}\sigma_p^i\sigma_n^j) \tag{1.4}$$

Here $\mathbf{P_p}$ and $\mathbf{P_n}$ are the polarization of the protons and neutrons respectively and C_{ij} is the correlation matrix. Suppose for example that the proton and neutron are in a spin singlet state. Proton and neutron polarization are both zero. But the spins are perfectly anti-correlated so that $C_{ij} = -\delta_{ij}$, that is the neutron is always spin up(down) if the proton is spin down(up).

Let's compute the cross section for the case where the spins of protons and neutrons are random. Then $\rho_i = \frac{1}{4}$ and

$$\frac{d\sigma}{d\Omega} = \mathrm{Tr}\rho_i M^\dagger M = \frac{1}{4}\mathrm{Tr}\left(|f_s|^2P_s+|f_t|^2P_t\right)$$

where we use the fact that$P_sP_t = 0$ and $P_sP_s = P_s$, etc. Taking the trace

$$\frac{d\sigma}{d\Omega} = \frac{1}{4}\left(|f_s|^2+3|f_t|^2\right)$$

The final state spin density matrix is

$$\begin{aligned} \rho_f &= \frac{M\rho_i M^\dagger}{\mathrm{Tr}\rho_i M^\dagger M} \\ &= \frac{\frac{1}{4}(|f_s|^2P_s+|f_t|^2P_t)}{d\sigma/d\Omega} \\ &= \frac{\frac{1}{4}(|f_s|^2(1-\sigma_{\mathbf{n}}\cdot\sigma_{\mathbf{p}})+|f_t|^2(3+\sigma_{\mathbf{n}}\cdot\sigma_{\mathbf{p}}))}{4d\sigma/d\Omega} \\ &= \frac{\frac{1}{4}((|f_s|^2+3|f_t|^2)+(|f_t|^2-|f_s|^2)\sigma_{\mathbf{n}}\cdot\sigma_{\mathbf{p}}))}{4d\sigma/d\Omega} \end{aligned}$$

Comparison with the general form of the density matrix in Equation **??** yields $P_s = P_t = 0$, and

$$C_{ij} = \frac{\frac{1}{4}(|f_t|^2 - |f_s|^2)\delta_{ij})}{4d\sigma/d\Omega}$$

If the scattering amplitude for the triplet state is greater(less) than for the singlet state, the spins of the scattered particles will be correlated (anticorrelated).