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Lecture XVII

Proton Neutron Scattering

Protons and neutrons are both spin 1/2. We will need to extend our scattering matrix to 4 dimensions to include all the possible spin combinations fof the two particles. A proton and neutron exist in a bound state as a deuteron with zero orbital angular momentum and one unit of total angular momentum. $J = 1$. There is no $j = 0$ or $l = 1$ bound state. Therefore the forces between the proton and neutron when the spins are aligned, is evidently different than when the spins are opposite. At low energy, scattering is all s-wave and there will be no mixing of the single and triplet states. We then write the scattering potential as

$$V = V_s + V_t$$

and the scattering matrix becomes

$$M = f_s(\theta)P_s + f_t(\theta)P_t$$

where P_s and P_t are the projection matrices for single and triplet states respectively.

1.1.1 Projection operators

We can determine the projection matrices by noting that the total angular momentum of two spin $\frac{1}{2}$ states is

$$\mathbf{J} = \frac{1}{2}(\sigma_{\mathbf{p}} + \sigma_{\mathbf{n}})$$

and that

$$\mathbf{J^2}|\ \rangle = j(j+1)|\ \rangle$$

For a singlet state, $\mathbf{J^2}|\ s\rangle = 0$ and for a triplet state $\mathbf{J^2}|\ t\rangle = 2$ and for the state that is some mixture of triplet and singlet

$$|\ \psi\rangle = a|\ t\rangle + b|\ s\rangle,$$

the operator $\mathbf{J^2}$ projects out the triplet component,

$$J^2|\ \psi\rangle = 2|\ t\rangle$$

Therefore

$$\begin{aligned} P_t &= N\mathbf{J^2} = N\left(\frac{1}{2}(\sigma_{\mathbf{p}} + \sigma_{\mathbf{n}})\right)^2 \\ &= N\frac{1}{4}({\sigma_{\mathbf{n}}}^2 + {\sigma_{\mathbf{p}}}^2 + 2\sigma_{\mathbf{n}}\cdot\sigma_{\mathbf{p}}) \\ &= N\tfrac{2}{4}(3 + \sigma_{\mathbf{n}}\cdot\sigma_{\mathbf{p}}) \end{aligned}$$

The normalization is determined by requiring

$$\begin{aligned}
P_t &= P_t^2 = N^2\frac{1}{4}(9+6\sigma_{\mathbf{n}}\cdot\sigma_{\mathbf{p}}+(\sigma_{\mathbf{n}}\cdot\sigma_{\mathbf{p}})^2) \\
&= N^2\frac{1}{4}(9+6\sigma_{\mathbf{n}}\cdot\sigma_{\mathbf{p}}+(3-2\sigma_{\mathbf{n}}\cdot\sigma_{\mathbf{p}})) \\
&= N^2\frac{1}{4}(12+4\sigma_{\mathbf{n}}\cdot\sigma_{\mathbf{p}}) \\
&= N^2(3+\sigma_{\mathbf{n}}\cdot\sigma_{\mathbf{p}}) \\
\rightarrow N &= \frac{1}{4}
\end{aligned}$$

so that

$$P_t = \frac{1}{4}(3+\sigma_{\mathbf{p}}\cdot\sigma_{\mathbf{n}})$$

Then since $P_s + P_t = 1$ we find

$$P_s = \frac{1}{4}(1-\sigma_{\mathbf{p}}\cdot\sigma_{\mathbf{n}})$$

1.1.2 Density matrix and differential cross section

The differential cross section for neutron proton scattering is

$$\frac{d\sigma}{d\Omega} = \mathrm{Tr}\rho_i M^\dagger M$$

where ρ_i is the spin density matrix of the initial state. As each of the initial state proton and neutron can be in one of two spin states. Therefore density operator for the initial state is a 4X4 matrix. Remembering that the density operator is a Hermitian matrix with unit trace, there are 15 free parameters. We can write the spin density matrix as

$$\rho = \frac{1}{4}(1+\mathbf{P_p}\cdot\sigma_{\mathbf{p}}+\mathbf{P_n}\cdot\sigma_{\mathbf{n}}+\sum_{ij}C_{ij}\sigma_p^i\sigma_n^j) \tag{1.1}$$

Here $\mathbf{P_p}$ and $\mathbf{P_n}$ are the polarization of the protons and neutrons respectively and C_{ij} is the correlation matrix. Suppose for example that the proton and neutron are in a spin singlet state. Proton and neutron polarization are both zero. But the spins are perfectly anti-correlated so that $C_{ij} = -\delta_{ij}$, that is the neutron is always spin up(down) if the proton is spin down(up).

Let's compute the cross section for the case where the spins of protons and neutrons are random. Then $\rho_i = \frac{1}{4}$ and

$$\frac{d\sigma}{d\Omega} = \mathrm{Tr}\rho_i M^\dagger M = \frac{1}{4}\mathrm{Tr}\left(|f_s|^2 P_s + |f_t|^2 P_t\right)$$

where we use the fact that$P_s P_t = 0$ and $P_s P_s = P_s$, etc. Taking the trace

$$\frac{d\sigma}{d\Omega} = \frac{1}{4}\left(|f_s|^2 + 3|f_t|^2\right)$$

The final state spin density matrix is

$$\begin{aligned}\rho_f &= \frac{M\rho_i M^\dagger}{\mathrm{Tr}\rho_i M^\dagger M} \\ &= \frac{\frac{1}{4}(|f_s|^2 P_s + |f_t|^2 P_t)}{d\sigma/d\Omega} \\ &= \frac{\frac{1}{4}(|f_s|^2(1 - \sigma_{\mathbf{n}} \cdot \sigma_{\mathbf{p}}) + |f_t|^2(3 + \sigma_{\mathbf{n}} \cdot \sigma_{\mathbf{p}}))}{4d\sigma/d\Omega} \\ &= \frac{\frac{1}{4}((|f_s|^2 + 3|f_t|^2) + (|f_t|^2 - |f_s|^2)\sigma_{\mathbf{n}} \cdot \sigma_{\mathbf{p}}))}{4d\sigma/d\Omega}\end{aligned}$$

Comparison with the general form of the density matrix in Equation 1.4 yields $P_s = P_t = 0$, and

$$C_{ij} = \frac{\frac{1}{4}(|f_t|^2 - |f_s|^2)\delta_{ij})}{4d\sigma/d\Omega}$$

If the scattering amplitude for the triplet state is greater(less) than for the singlet state, the spins of the scattered particles will be correlated (anticorrelated).

1.2 Scattering identical particles

For spinless bosons the space-wave function solution to Schrodinger's equation is symmetric and will have the form

$$\langle x \mid \psi \rangle = e^{i\mathbf{k}\cdot\mathbf{x}} + e^{-i\mathbf{k}\cdot\mathbf{x}} + \frac{e^{ikr}}{r}(f(\theta) + f(\pi - \theta)) \tag{1.2}$$

where $\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2$ corresponding to particle 1 and 2. The differential cross section is

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= |f(\theta) + f(\pi - \theta)|^2 \\ &= |f(\theta)|^2 + |f(\pi - \theta)|^2 + 2\Re f^*(\theta) f(\pi - \theta) \end{aligned}$$

The "classical" cross section for two identical particles is given by just the first two terms. We will multiply by an overall $\frac{1}{2}$ to account for that. The third interference term is purely quantum mechanical and it will enhance the rate at $\theta = \pi/2$.

Consider scattering of Helium nuclei. The most common isotope, He^4 has zero spin. He^3 has spin $\frac{1}{2}$. The coulomb scattering, which dominates at low energy, is indifferent to the number of neutrons, so to a very good approximation, the amplitude for

$$\mathrm{He}^3 + \mathrm{He}^4 \to \mathrm{He}^3 + \mathrm{He}^4 \tag{1.3}$$

is the same as for

$$\mathrm{He}^4 + \mathrm{He}^4 \to \mathrm{He}^4 + \mathrm{He}^4 \tag{1.4}$$

The differential cross section for scattering of distinquishable particles (process **??**) is given by

$$\frac{d\sigma}{d\Omega} = |f(\theta) + f(\pi - \theta)|^2 = |f(\theta)|^2 + |f(\pi - \theta)|^2$$

and for identical spin zero particles

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 + |f(\pi - \theta)|^2 + 2\Re f^*(\theta) f(\pi - \theta)$$

Now suppose that we are scattering identical spin 1/2 particles, like electrons or He^3 nuclei. The particles can be in a singlet antisymmetric spin state and symmetric space wave function or a triplet symmetric spin state and antisymmetric wave function. The scattering matrix is 2X2 as there are four spin states to consider. There will be no mixing of singlet and triplet if parity is conserved since the space wave functions are symmetric and antisymmetric respectively. If the particles are distinguishable , like protons and neutrons,

$$M(\mathbf{k}', \mathbf{k}) = M_s(\mathbf{k}', \mathbf{k}) + M_t(\mathbf{k}', \mathbf{k})$$

The scattering matrix will then be written in terms of single and triplet separately.

$$M(\mathbf{k}, \mathbf{k}') = [M_s(\mathbf{k}, \mathbf{k}') + M_s(-\mathbf{k}', \mathbf{k})] + [M_t(\mathbf{k}, \mathbf{k}') - M_t(-\mathbf{k}', \mathbf{k})]$$

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \mathrm{Tr}\rho_i M^\dagger M \tag{1.5}$$

where ρ_i is the density matrix of the initial state and the density matrix of the final state is

$$\rho_f = \frac{M \rho_i M^\dagger}{d\sigma/d\Omega} \tag{1.6}$$

1.2.1 Identical fermions

Now back to identical particles. Again assuming no polarization in the initial state

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{1}{4}\mathrm{Tr}(M^\dagger M) \\
&= \frac{1}{4}\mathrm{Tr}\left([(f_s(k,\theta)+f_s(k,\pi-\theta))P_s + (f_t(k,\theta)-f_t(k,\pi-\theta))P_t]\right. \\
&\quad \left.\times[(f_s^*(k,\theta)+f_s^*(k,\pi-\theta))P_s + (f_t^*(k,\theta)-f_t^*(k,\pi-\theta))P_t]\right) \\
&= \frac{1}{4}\left(|f_s(k,\theta)+f_s(k,\pi-\theta)|^2 P_s + 3|f_t(k,\theta)-f_t(k,\pi-\theta)|^2 P_t\right) \\
&= \frac{1}{4}\left(|f_s(k,\theta)+f_s(k,\pi-\theta)|^2 + 3|f_t(k,\theta)-f_t(k,\pi-\theta)|^2\right)
\end{aligned}$$

The final state density matrix is

$$\begin{aligned}
\rho_f &= \frac{1}{4}\frac{([f_s(\theta)+f_s(\pi-\theta))P_s + (f_t(\theta)-f_t(\pi-\theta))P_t]\,[(f_s^*(\theta)+f_s^*(\pi-\theta))P_s + (f_t^*(\theta)-f_t^*(\pi-\theta))P_t]}{d\sigma/d\Omega} \\
&= \frac{1}{4}\frac{|f_s(\theta)+f_s(\pi-\theta)|^2 P_s + |f_t(\theta)-f_t(\pi-\theta)|^2 P_t}{4d\sigma/d\Omega} \\
&= \frac{1}{4}\frac{|f_s(\theta)+f_s(\pi-\theta)|^2(1-\boldsymbol{\sigma_1}\cdot\boldsymbol{\sigma_2}) + |f_t(\theta)-f_t(\pi-\theta)|^2(3+\boldsymbol{\sigma_1}\cdot\boldsymbol{\sigma_2})}{4d\sigma/d\Omega} \\
&= \frac{1}{4}\frac{|f_s(\theta)+f_s(\pi-\theta)|^2 + 3|f_t(\theta)-f_t(\pi-\theta)|^2 + |f_t(\theta)+f_t(\pi-\theta)|^2 - |f_s(\theta)-f_s(\pi-\theta)|^2\boldsymbol{\sigma_1}\cdot\boldsymbol{\sigma_2})}{4d\sigma/d\Omega}
\end{aligned}$$

If the amplitudes for scattering in singlet and triplet states are the same, that is if the forces are spin independent (no $\boldsymbol{\sigma_1}\cdot\boldsymbol{\sigma_2}$ term in the potential and $f_s = f_t$), then

$$\frac{d\sigma}{d\Omega} = |f(k,\theta)|^2 + f(k,\pi-\theta)|^2 - \Re f^*(k,\theta)f(k,\pi-\theta) \tag{1.7}$$

There is evidently no polarization in the final state density matrix, but there is a correlation and the correlation is

$$C_{ij} = \frac{1}{4}\frac{|f_t(\theta)+f_t(\pi-\theta)|^2 - |f_s(\theta)-f_s(\pi-\theta)|^2}{4d\sigma/d\Omega} \tag{1.8}$$

and if $f_s = f_t$

$$C_{ij} = -\delta_{ij}\frac{\Re f(k,\theta)f^*(k,\pi-\theta)}{d\sigma/d\Omega} \tag{1.9}$$

Unlike proton neutron scattering (distinquishable particles), there is a correlation even when the interaction has no spin dependence. If there is a spin dependent interaction

$$V(r) = V_0(r) + V_1(r)\boldsymbol{\sigma_1}\cdot\boldsymbol{\sigma_2}$$

then the correlation is given by

$$C_{ij} = \delta_{ij}\frac{|f_t(\theta)-f_t(\pi-\theta)|^2 - |f_s(\theta)-f_s(\pi-\theta)|^2}{3|f_t(\theta)-f_t(\pi-\theta)|^2 + |f_s(\theta)+f_s(\pi-\theta)|^2} \tag{1.10}$$

At $\theta = \pi/2$, there is perfect correlation and $C_{ij} = -\delta_{ij}$.