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Lecture XVIII

Quantization of the E-M field

2.1 Classical E&M

First we will work in the transverse gauge where there are no sources. Then $\nabla \cdot \mathbf{A} = 0$, $\nabla \times \mathbf{A} = \mathbf{B}$, and $\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$ and Maxwell's equations are

$$\begin{aligned} \nabla \times \mathbf{B} &= \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \\ \nabla \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{E} &= 0 \\ \nabla \cdot \mathbf{B} &= 0 \end{aligned}$$

Combining the above gives a wave equation for the vector potential

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = 0$$

Start with periodic boundary conditions. Then there are plane wave solutions

$$e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

where

$$\mathbf{k} = \frac{2\pi}{L}(n_x\hat{x} + n_y\hat{y}_n + n_z\hat{z})$$

The plane waves form a complete orthonormal set.

$$\begin{aligned} \frac{1}{V}\int_V d\mathbf{r}\; e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} &= \delta_{kk'} \\ \frac{1}{V}\sum_k e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} &= \delta(\mathbf{r}-\mathbf{r}') \end{aligned}$$

To go from periodi boundary conditions to a continuous spectrum

$$\frac{1}{V}\sum_k \to \int \frac{d\mathbf{k}}{(2\pi)^3}, \quad \frac{V}{(2\pi)^3}\delta_{kk'} \to \delta(\mathbf{k}-\mathbf{k}')$$

The expansion of the vector potential as plane waves is

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{\sqrt{V}}\sum_k [e^{i\mathbf{k}\cdot\mathbf{r}}\mathbf{A}_{\mathbf{k}}(\mathbf{t}) + c.c.]$$

$\mathbf{A}$ is real. The transverse gauge conditions $\nabla \cdot \mathbf{A} = 0 \to \mathbf{k}\cdot\mathbf{A} = 0$. We get the time dependence from the wave equation

$$\mathbf{A}_k(t) = \mathbf{A}_{\mathbf{k}}e^{-i\omega t}$$

Then

$$\begin{aligned}
\mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = \frac{i}{c\sqrt{V}}\sum_k \omega[e^{i\mathbf{k}\cdot\mathbf{r}}\mathbf{A}_{\mathbf{k}}(t) - c.c.] \\
\mathbf{B} &= \nabla \times \mathbf{A} = \frac{i}{\sqrt{V}}\sum_k [e^{i\mathbf{k}\cdot\mathbf{r}}\mathbf{k} \times \mathbf{A}_{\mathbf{k}}(t) - c.c.]
\end{aligned}$$

2.1.1 Polarization

We need a mutually orthogonal pair of polarization vectors. We can use the real orthogonal unit vectors $\epsilon_{\mathbf{i}}$, $i = 1, 2$ with

$$\epsilon_{\mathbf{1}} \times \epsilon_{\mathbf{2}} = \hat{\mathbf{k}}, \quad \epsilon_{k\alpha} \cdot \epsilon_{k\beta} = \delta_{\alpha\beta}$$

Or we can use complex circular polarization vectors

$$\mathbf{e}_{k\pm 1} = \mp\frac{1}{\sqrt{2}}(\epsilon_{k1} \pm i\epsilon_{k2})$$

The total energy is

$$H = \frac{1}{2}\int d^3\mathbf{r}(E^2 + B^2)$$

and using the expansions of E and B in terms of A we can write the energy in terms of A as

$$H = 2\sum_k k^2|\mathbf{A}_k(t)|^2$$

We want to quantize the fields. The usual strategy is to identify canonically conjugate variables, assign them operator status that obey the canonical commutation rule. Working backwards towards what we learned when quantizing the harmonic oscillator. Indeed recognizing that plane waves that represent $\mathbf{A}$ are a solution to the harmonic oscillator equation, we define

$$\mathbf{Q}_k(t) = \frac{1}{c}[\mathbf{A}_k(t) + c.c.], \qquad \mathbf{P}_k(t) = -ik[\mathbf{A}_k(t) - c.c.]$$

We can rewrite

$$H = \frac{1}{2}\sum_k(\mathbf{P}_k^2 + \omega^2\mathbf{Q}_k^2).$$

and hamilton's equations

$$\dot{\mathbf{P}}_k = -\frac{\partial H}{\partial \mathbf{Q}_{\mathbf{k}}}, \quad \dot{\mathbf{Q}}_k = \frac{\partial H}{\partial \mathbf{P}_{\mathbf{k}}}$$

would allow us to work from Hamiltonian to equations of motion for $\mathbf{A}$.

2.2 Quantization

We raise the canonical variables to operator status and the commutation rules are

$$[Q_{k\lambda}, P_{k'\lambda'}] = i\hbar\delta_{kk'}\delta_{\lambda\lambda'}, \quad [Q_{k\lambda}, Q_{k'\lambda'}] = 0, \quad [P_{k\lambda}, P_{k'\lambda'}] = 0$$

Next define annihilation and creation operators

$$a_{k\lambda} = \frac{1}{\sqrt{2\hbar\omega}}(\omega Q_{k\lambda} + iP_{k\lambda})$$

and $a^\dagger$. We find

$$[a_{k\lambda}, a^\dagger_{k'\lambda'}] = \delta_{kk'}\delta_{\lambda\lambda'}$$

Define the vector operator $\mathbf{a}_k = a_{k1}\mathbf{e_1} + a_{k2}\mathbf{e_2}$ or $a_{k-1}\mathbf{e}_- + a_{k+1}\mathbf{e}_+$. Then

$$H = \sum_k \hbar\omega_k[\mathbf{a}^\dagger_\mathbf{k} \cdot \mathbf{a_k} + \frac{1}{2}] = \sum_{k\lambda} \hbar\omega_k[N_{k\lambda} + \frac{1}{2}],$$

In the heisenberg picture the equations of motion for $\mathbf{a}_k$ are

$$i\hbar\dot{\mathbf{a}}_k(t) = [\mathbf{a}_k, H] = \hbar\omega_k\mathbf{a}_k(t)$$

The operators $\mathbf{a_k}$ are the quantum analog of the fourier coefficients $\mathbf{A}$. Getting the units right we assign

$$\mathbf{A} \to \sqrt{\hbar c/2k}\mathbf{a}_k(t).$$

Finally write the field operators in terms of $\mathbf{a}_k$.

$$\begin{aligned}
\mathbf{A} &= \sum_k \sqrt{\frac{\hbar c}{2Vk}}[e^{i(\mathbf{k}\cdot\mathbf{r})-\omega t}\mathbf{a}_k + h.c.] \\
\mathbf{E} &= i\sum_k \sqrt{\frac{\hbar ck}{2V}}[e^{i\mathbf{k}\cdot\mathbf{r}-\omega t}\mathbf{a_k} - h.c.] \\
\mathbf{B} &= i\sum_k \sqrt{\frac{\hbar c}{2Vk}}[e^{i\mathbf{k}\cdot\mathbf{r}-\omega t}\mathbf{k}\times\mathbf{a_k} - h.c.]
\end{aligned}$$

Heisenberg equations of motion for the field operators give Maxwell's equations. Momentum operator is constructed from field operators. Hermitian version is

$$\mathbf{P} = \frac{1}{2c}\int d\mathbf{r}[\mathbf{E}\times\mathbf{B} - \mathbf{B}\times\mathbf{E}] = \sum_{k\lambda}\hbar\mathbf{k}N_{k\lambda},$$

The momentum operator is originally constructed as the generator of translations. Let's check it out. The translation operator is

$$T(\mathbf{d}) = e^{-i\mathbf{d}\cdot\mathbf{P}/\hbar}$$

Consider translation of the efield.

$$T^\dagger(\mathbf{d})\mathbf{E}(\mathbf{r}, t)T(\mathbf{d}) = \mathbf{E}(\mathbf{r} - \mathbf{d}, t)$$

An infinitesmal translation would have the form

$$(1 + i\mathbf{d}\cdot\mathbf{P}/\hbar)\mathbf{E}(\mathbf{r},t)(1 - i\mathbf{d}\cdot\mathbf{P}/\hbar) = \mathbf{E}(\mathbf{r}) - \mathbf{d}\cdot\frac{\partial\mathbf{E}}{\partial\mathbf{r}}$$

Note that the expectaion value of E or B in a state with a definite number of photons is zero. A coherent superposition of single photon states that satisfies

$$a_k|\,z\rangle = z|\,z\rangle$$

is

$$|\,z\rangle = e^{-\frac{1}{2}|z|^2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|\,\rangle n$$

has nonzero expectation value for electric and magnetic fields and represents a classical field.

The angular momentum in the classical field is

$$\mathbf{J}_{cl} = \frac{1}{c}\int d\mathbf{r}[\mathbf{r}\times(\mathbf{E}\times\mathbf{B})]_{cl}$$

which can be separated into intrinsic angular momentum (spin) that does not depend on the position of the origin

$$\mathbf{J}_{spin} = \frac{1}{2c}\int d\mathbf{r}(\mathbf{E}\times\mathbf{A} - \mathbf{A}\times\mathbf{E}),$$

and the part that does

$$\mathbf{J}_{orb} = \frac{1}{2c}\int d\mathbf{r}\sum_{i=1}^{3}[E_i(\mathbf{r}\times\nabla A_i) + (\mathbf{r}\times\nabla A_i)E_i].$$

The spin part in terms of operators becomes

$$\mathbf{J}_{spin} = -i\hbar\sum_k \mathbf{a}_\mathbf{k}^\dagger\times\mathbf{a}_\mathbf{k} = \hbar\sum_{k\lambda}\lambda\hat{\mathbf{k}}n_{k\lambda}$$

The one photon state has helicity $\lambda = \pm 1$.

The relation between single photon states and states with definite linear and angular momentum is given by

$$\begin{aligned}
|\,kjm\lambda\rangle &= \sqrt{\frac{2j+1}{4\pi}}\int d\mathbf{n}D^j_{m\lambda}(\mathbf{n})^* a_\lambda^\dagger(\mathbf{k})|\,0\rangle, \\
a^\dagger(\mathbf{k})|\,0\rangle &= \sum_{j=1}^{\infty}\sum_{m=-1}^{j}\sqrt{\frac{2j+1}{4\pi}}D^j_{m\lambda}(\mathbf{n})|\,kjm\lambda\rangle
\end{aligned}$$

Note, the sum starts at 1. There is no zero helicty single photon state so there is no $j = 0$ contribution to angular momentum.