2.1 Quantization

We raise the canonical variables to operator status and the commutation rules are

\[ [Q_{k\lambda}, P_{k'\lambda'}] = i\hbar \delta_{kk'} \delta_{\lambda\lambda'}, \quad [Q_{k\lambda}, Q_{k'\lambda'}] = 0, \quad [P_{k\lambda}, P_{k'\lambda'}] = 0 \]

Next define annihilation and creation operators

\[ a_{k\lambda} = \frac{1}{\sqrt{2\hbar}} (\omega Q_{k\lambda} + iP_{k\lambda}) \]

and \( a^\dagger \). We find

\[ [a_{k\lambda}, a_{k'\lambda'}^\dagger] = \delta_{kk'} \delta_{\lambda\lambda'} \]

Define the vector operator \( a_k = a_{k1} e_1 + a_{k2} e_2 \) or \( a_{k-1} e_- + a_{k+1} e_+ \). Then

\[ H = \sum_k \hbar \omega_k [a_k^\dagger \cdot a_k + \frac{1}{2}] = \sum_{k\lambda} \hbar \omega_k [N_{k\lambda} + \frac{1}{2}] \]

In the heisenberg picture the equations of motion for \( a_k \) are

\[ i\hbar \dot{a}_k(t) = [a_k, H] = \hbar \omega_k a_k(t) \]

The operators \( a_k \) are the quantum analog of the fourier coefficients \( A \). Getting the units right we assign

\[ A \rightarrow \sqrt{\hbar c/2} a_k(t). \]

Finally write the field operators in terms of \( a_k \).

\[
\begin{align*}
A &= \sum_k \sqrt{\frac{\hbar c}{2Vk}} [e^{i(k \cdot r - \omega t)} a_k + h.c.] \\
E &= i \sum_k \sqrt{\frac{\hbar ck}{2V}} [e^{i(k \cdot r - \omega t)} a_k - h.c.] \\
B &= i \sum_k \sqrt{\frac{\hbar c}{2V}} [e^{i(k \cdot r - \omega t)} \mathbf{k} \times a_k - h.c.]
\end{align*}
\]

The heisenberg equations of motion for the field operators give Maxwell’s equations. Momentum operator is constructed from field operators. Hermitian version is

\[ \mathbf{P} = \frac{1}{2c} \int d^3r [\mathbf{E} \times \mathbf{B} - \mathbf{B} \times \mathbf{E}] = \sum_{k\lambda} \hbar k N_{k\lambda}, \]
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The momentum operator is originally constructed as the generator of translations. Let’s check it out. The translation operator is

\[ T(d) = e^{-id \cdot P/\hbar} \]

Consider translation of the efield.

\[ T^\dagger(d)E(r,t)T(d) = E(r - d, t) \]

An infinitesimal translation would have the form

\[(1 + i d \cdot P/\hbar)E(r,t)(1 - i d \cdot P/\hbar) = E(r) - d \frac{\partial E}{\partial r} \]

Note that the expectation value of \( E \) or \( B \) in a state with a definite number of photons is zero. A coherent superposition of single photon states that satisfies

\[ a_k | z \rangle = z | z \rangle \]

is

\[ | z \rangle = e^{-\frac{i}{2}|z|^2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} | n \rangle \]

has nonzero expectation value for electric and magnetic fields and represents a classical field.

The angular momentum in the classical field is

\[ J_{cl} = \frac{1}{c} \int dr [r \times (E \times B)]_{cl} \]

which can be separated into intrinsic angular momentum (spin) that does not depend on the position of the origin

\[ J_{spin} = \frac{1}{2c} \int dr (E \times A - A \times E), \]

and the part that does

\[ J_{orb} = \frac{1}{2c} \int dr \sum_{i=1}^{3} [E_i(r \times \nabla A_i) + (r \times \nabla A_i)E_i]. \]

The spin part in terms of operators becomes

\[ J_{spin} = -i\hbar \sum_k a_k^\dagger \times a_k = \hbar \sum_{k\lambda} \lambda \hbar n_{k\lambda} \]

The one photon state has helicity \( \lambda = \pm 1 \).

The relation between single photon states and states with definite linear and angular momentum is given by

\[ | kjm\lambda \rangle = \sqrt{\frac{2j+1}{4\pi}} \int d\mathbf{n} D_{m\lambda}^j(\mathbf{n})^* a^\dagger_{\lambda}(\mathbf{k}) | 0 \rangle, \]

\[ a^\dagger(\mathbf{k}) | 0 \rangle = \sum_{j=1}^{\infty} \sum_{m=-1}^{j} \sqrt{\frac{2j+1}{4\pi}} D_{m\lambda}^j(\mathbf{n}) | kjm\lambda \rangle \]

Note, the sum starts at 1. There is no zero helicity single photon state so there is no \( j = 0 \) contribution to angular momentum.
2.1. QUANTIZATION

2.1.1 Parity and Time Reversal

On space inversion, \( \mathbf{E}(r, t) \rightarrow -\mathbf{E}(-r, t) \). Think about a pair of capacitor plates at \( z = \pm a \). Reflection in the x-y plane reverses \( \mathbf{E} \). Then since \( \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \), it must be that under inversion \( \mathbf{A}(r, t) \rightarrow -\mathbf{A}(-r, t) \). And since \( \mathbf{B} = \nabla \times \mathbf{A} \), under inversion \( \mathbf{B}(r, t) \rightarrow \mathbf{B}(-r, t) \). To determine how the creation and annihilation operator transform consider

\[
\mathbf{A}(r, t) = \sqrt{\hbar c} \int \frac{d^3k}{(2\pi)^3 2k} \sum_\lambda [e^{i(kr)}e_\lambda(k)a_\lambda(k, t) + h.c.] \tag{2.1}
\]

\[
P^1 \mathbf{A}(r, t)P = \sqrt{\hbar c} \int \frac{d^3k}{(2\pi)^3 2k} \sum_\lambda [e^{i(kr)}e_\lambda(k)P^1a_\lambda(k, t)P + h.c.] \tag{2.2}
\]

\[
-\mathbf{A}(-r, t) = -\sqrt{\hbar c} \int \frac{d^3k}{(2\pi)^3 2k} \sum_\lambda [e^{i(-kr)}e_\lambda(k)a_\lambda(k, t) + h.c.] \tag{2.3}
\]

Then in the last \( k \rightarrow -k \),

\[
-\mathbf{A}(-r, t) = -\sqrt{\hbar c} \int \frac{d^3k}{(2\pi)^3 2k} \sum_\lambda [e^{i(-kr)}e_\lambda(-k)a_\lambda(-k, t) + h.c.] \tag{2.5}
\]

\[
= -\sqrt{\hbar c} \int \frac{d^3k}{(2\pi)^3 2k} \sum_\lambda [e^{i(kr)}e_{-\lambda}(k)a_{-\lambda}(-k, t) + h.c.] \tag{2.6}
\]

\[
= -\sqrt{\hbar c} \int \frac{d^3k}{(2\pi)^3 2k} \sum_\lambda [e^{i(kr)}e_\lambda(k)a_{-\lambda}(-k, t) + h.c.] \tag{2.7}
\]

Comparing Equations 2.2 and 2.7 we find that

\[
P^1a_\lambda(k, t)P = -a_{-\lambda}(-k, t) \tag{2.8}
\]

Evidently the intrinsic parity of a photon is odd. Like the electric field. Think about the emission of a photon by an atom that transitions from \( l \) to \( l \pm 1 \), changing the sign of its parity. Similarly

\[
\mathbf{A}(r, t) = \sqrt{\hbar c} \int \frac{dk}{(2\pi)^3 2k} \sum_\lambda [e^{i(kr)}e_\lambda(k)a_\lambda(k, t) + h.c.] \tag{2.9}
\]

\[
I_t \mathbf{A}(r, t)I_t^{-1} = -\mathbf{A}(r, -t) = -\sqrt{\hbar c} \int \frac{dk}{(2\pi)^3 2k} \sum_\lambda [e^{i(kr)}e_\lambda(k)a_\lambda(k, -t) + h.c.] \tag{2.10}
\]

\[
= \sqrt{\hbar c} \int \frac{dk}{(2\pi)^3 2k} \sum_\lambda [e^{-i(kr)}e^*_\lambda(k)I_ra_\lambda(k, t)I_t^{-1} + h.c.] \tag{2.11}
\]

\[
= -\sqrt{\hbar c} \int \frac{dk}{(2\pi)^3 2k} \sum_\lambda [e^{-i(kr)}e_{-\lambda}(k)I_ra_\lambda(k, t)I_t^{-1} + h.c.] \tag{2.12}
\]

\[
= -\sqrt{\hbar c} \int \frac{dk}{(2\pi)^3 2k} \sum_\lambda [e^{i(kr)}e_\lambda(k)I_ra_\lambda(-k, t)I_t^{-1} + h.c.] \tag{2.13}
\]
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Then Equation 2.10 is the same as Equation 2.13 if $I_t a_\lambda(k, t) I_t^{-1} = a_\lambda(-k, -t)$. In the last two steps we use $e_\lambda^a(k) = -e_{-\lambda}(k)$ and then $e_\lambda(-k) = e_{-\lambda}(k)$.

**Higgs spin from \( \rightarrow \gamma \gamma \)**

Let’s assume that Parity is conserved, that the initial state and final state both have definite parity. A single photon state with definite helicity $a_\lambda^1 | 0 \rangle$ has odd parity. That is

$$P | k, \lambda \rangle = -| -k, -\lambda \rangle$$

It turns out that particle and anti-particle always have opposite parity. Therefore, a positronium $s$-state has odd parity. It can decay to two photons, and since parity is conserved, the two photon final state also has odd parity. If the final state is two photons with definite helicity, the parity is even. In general the effect of the parity operator on a two photon state is

$$P | \rangle \rangle = P | k, \lambda \rangle_1 | -k, -\lambda \rangle_2$$

However we can construct a final state that is not simply a product of definite helicity photon states. Suppose for example our two photon state

$$| \gamma \gamma \rangle = \frac{1}{2} ( | k, \lambda \rangle_1 | -k, -\lambda \rangle_2 + | -k, \lambda \rangle_2 | k, -\lambda \rangle_1 $$

$$P | \rangle \rangle = \frac{1}{2} ( | -k, \lambda \rangle_1 | k, -\lambda \rangle_2 $$

Rotate $180^\circ$ so that $k \rightarrow -k$. Then

$$RP | \rangle \rangle = \frac{1}{2} ( | k, -\lambda \rangle_1 | -k, -\lambda \rangle_2 + | -k, \lambda \rangle_2 | k, -\lambda \rangle_1 $$

The two photon state with linear combinations of helicity states corresponding to perpendicular linear polarization states has odd parity. The two photon state with parallel linear polarization states has even parity. $\pi^0$ is quark antiquark, has odd parity and decays to two photons with perpendicular linear polarization. The Higgs has spin parity $0^+$. So if we can measure the photon polarization we can determine the parity. But that does not quite give us the total spin. We need to assume that there is some preferred helicity in the intermediate state that decays to the Higgs to determine the spin.
2.2 Commutation Rules for Field Operators

As the field operators correspond to local observables, (defined at each space time point), the operators in general do not commute and there is an associated uncertainty in measurement of the fields. Since we know how to write the fields in terms of the creation and annihilation operators, and we know how they commute, we can develop the commutation rules of the fields themselves. Beginning with the vector potential,

\[ A_i(x_1) = \sqrt{\hbar} c \int \frac{d^3k}{\sqrt{2k(2\pi)^3}} \sum_{\lambda} \left( \epsilon_{\lambda i}^k(k) a_{\lambda}(k) e^{i k \cdot x_1} + h.c. \right) \]

at two distinct space time points \( x_1 \) and \( x_2 \),

\[ [A_i(x_1), A_j(x_2)] = \hbar c \int \frac{d\mathbf{k}}{2k(2\pi)^3} [e^{i \mathbf{k}(x_1-x_2)} t_{ij} - c.c.] \]  \hfill (2.14)

\[ = \hbar c \int \frac{d\mathbf{k}}{2k(2\pi)^3} [e^{i \mathbf{k}(r_1-r_2)} e^{i k|t_1-t_2|} t_{ij} - c.c.] \]  \hfill (2.15)

where

\[ t_{ij} = \sum_{\lambda} (\epsilon_{\lambda i}^k)^* \epsilon_{\lambda j}^k \]

and

\[ [a_{\lambda}(k), a^\dagger_{\lambda'}(k)] = \delta_{\lambda\lambda'} \delta(k-k') \]

Now consider \( t_{ij} \). Think about the cartesian representation of the polarization unit vectors, \( e_\alpha \). Because the polarization is transverse to \( \mathbf{k} \), \( \alpha = 1, 2 \) and not 3. If the sum were \( \alpha = 1, 2, 3 \) then \( \sum_\alpha \epsilon_{\alpha i} \) would be the projection of a unit vector onto \( i \) and it would be perpendicular to the projection of that same unit vector onto \( j \) and we would have that \( \sum_\alpha (\epsilon_{\alpha i})^* \epsilon_{\alpha j} = \delta_{ij} \). But \( \alpha \) only runs from 1 to 2. The 3 component is in the \( \mathbf{k} \) direction. Therefore

\[ t_{ij} = \delta_{ij} - \frac{k_i k_j}{k^2} \]

The commutator for the electric field is

\[ [E_i(x_1), E_j(x_2)] = \frac{1}{c^2} \left[ \frac{\partial A_i(x_1)}{\partial t_1} - \frac{\partial A_j(x_2)}{\partial t_2} \right] = \frac{1}{c^2} \frac{\partial}{\partial t_1} \frac{\partial}{\partial t_2} [A_i(x_1), A_j(x_2)] \]

Since

\[ \frac{\partial}{\partial t_1} \frac{\partial}{\partial t_2} e^{i(k \cdot r-|k||t|)} = -k^2 e^{i(k \cdot r-|k||t|)} \]

and

\[ \frac{\partial}{\partial r_{1i}} \frac{\partial}{\partial r_{2j}} e^{i(k \cdot r-|k||t|)} = -k_i k_j e^{i(k \cdot r-|k||t|)} \]

we can write

\[ \frac{k_i k_j}{k^2 c^2} \frac{\partial}{\partial t_1} \frac{\partial}{\partial t_2} e^{i(k \cdot r-|k||t|)} = \frac{\partial}{\partial r_{1i}} \frac{\partial}{\partial r_{2j}} e^{i(k \cdot r-|k||t|)} \]

Then

\[ [E_i(x_1), E_j(x_2)] = 2i \hbar c \left( \delta_{ij} \frac{1}{c^2} \frac{\partial}{\partial t_1} \frac{\partial}{\partial t_2} - \frac{\partial}{\partial r_{1i}} \frac{\partial}{\partial r_{2j}} \right) D(r,t) \]
2.2. COMMUTATION RULES FOR FIELD OPERATORS

where

\[ D(r, t) = -\int \frac{d^3k}{2k(2\pi)^3} \left[ e^{i(k \cdot r - |k|ct)} - c.c. \right] = \frac{1}{8\pi r} \left[ \delta(r + ct) - \delta(r - ct) \right]. \]

Field operators evidently commute if localized to points in space time that cannot be connected by a light signal.

We find that

\[ [B_i(x_1), B_j(x_2)] = [E_i(x_1), E_j(x_2)] \]
\[ [E_i(x_1), B_i(x_2)] = 0 \]
\[ [E_i(x_1), B_j(x_2)] = -2i\hbar \epsilon_{ijk} \frac{\partial}{\partial t_1} \frac{\partial}{\partial r_{2k}} D(r, t) \]

At equal times \( t_1 = t_2 \),

\[ [E_i(r_1, t), E_j(r_2, t)] = [B_i(r_1, t), B_j(r_2, t)] = 0 \]

but

\[ [E_i(r_1, t), B_j(r_2, t)] \neq 0 \]

so electric and magnetic fields cannot be specified simultaneously at all points in space.
2.3 Uncertainty Relations with EM fields

Uncertainty relations are related to commutators according to

\[ \Delta V_1 \Delta V_2 \geq \frac{1}{2} |\langle [V_1, V_2] \rangle|, \]

where \( V_1 \) and \( V_2 \) are operators. Suppose that we have two small regions of space-time volumes \( \Omega_1 \) and \( \Omega_2 \) respectively and that \( \Omega_2 \) is in the future with respect to \( \Omega_1 \). \( (T_1 \) and \( T_2 \) are the extent in time of each region, \( T_1 = t'_1 - t_1 \), etc. The field averaged over the volume is

\[ E(\Omega) = \frac{1}{\Omega} \int_\Omega d\Omega E(x) \]

The commutator for perpendicular (non-parallel) components of \( E \) in regions \( \Omega_1 \) and \( \Omega_2 \) is

\[
\begin{align*}
[E_i(x_1), E_j(x_2)] &= 2i\hbar c \left( \frac{1}{v^2} \frac{\partial^2}{\partial t_1 \partial t_2} - \frac{\partial^2}{\partial r_1 \partial r_2} \right) D(r,t) \quad (2.16) \\
&\rightarrow = -2i\hbar c \left( \frac{\partial^2}{\partial r_1 \partial r_2} \right) D(r,t) \quad (2.17) \\
&= -\frac{2}{8\pi} i\hbar c \frac{\partial^2}{\partial x_1 \partial y_2} \frac{\delta(r-ct)}{r} \quad (2.18) \\
\rightarrow \Delta E_x(\Omega_1) \Delta E_y(\Omega_2) \geq \frac{\hbar c}{8\pi} \int_{\Omega_1} d\Omega_1 \int_{\Omega_2} d\Omega_2 \frac{\partial^2}{\partial x_1 \partial y_2} \frac{\delta(r-ct)}{r} \quad (2.19)
\end{align*}
\]

Very interesting. We evidently cannot determine orthogonal components of the electric field that can be connected by a light signal with arbitrary precision. The product of the uncertainties does however decrease with the spatial separation of the points. Which makes sense. We don’t expect a very distant disturbance to have much of an effect locally.

The uncertainty in fields in Equation 2.19 follows from the construction of vector potential and then field operators in terms of creation and annihilation operators, and the commutator of those operators that follows from their association with canonical variables that behaved like \( P \) and \( Q \) and where hamilton’s equations were equivalent to Maxwell’s equations. We have not connected them in any way with uncertainty in real momentum and position space. That is the next step. We attempt to determine the electric field along the \( x \)-direction in volume \( \Omega_1 \) by measuring the change in the momentum of a charge that is accelerated across the volume. Because of the fundamental limit on how well we can measure momentum, there is a limit on how well we measure the electric field. Meanwhile, the test charge induces a scalar and vector potential in volume \( \Omega_2 \). There will be some uncertainty in the fields in region \( \Omega_2 \) since we are not sure where precisely the test charge is located or how fast it is moving. The product of the uncertainties of the fields in the two regions is an independent check on the consistency of the quantization formalism.

First we determine the \( x \)-component of the E-field in region \( \Omega_1 \) with test charge \( Q \).

\[
\begin{align*}
E_x(\Omega_1) &\sim \frac{p_x(t'_1) - p_x(t_1)}{Q(t'_1 - t_1)} \\
\rightarrow \Delta E_x(\Omega_1) &\geq \frac{\hbar}{2QT_1 \Delta x}
\end{align*}
\]
where the uncertainty in the change in momentum $p$ is related to the uncertainty in position $\Delta x$. In region $\Omega_2$ there is a scalar and vector potential associated with $Q$ in region $\Omega_1$. The scalar potential due to the charge is

$$
\phi(r_2, t_2) = \int_{\Omega_1} dr_1 cdt_1 \rho(r_1, t) \frac{\delta(c(t_2 - t_1) - |r_2 - r_1|)}{4\pi |r_1 - r_2|} \\
= \frac{Q}{V_1} \int_{\Omega_1} dr_1 cdt_1 \frac{\delta(c(t_2 - t_1) - |r_2 - r_1|)}{4\pi |r_1 - r_2|} \\
\rightarrow \Delta \phi(r_2, t_2) = \frac{cQ\Delta x}{V_1} \int_{\Omega_1} dr_1 cdt_1 \frac{\partial}{\partial x_1} \frac{\delta(c(t_2 - t_1) - |r_2 - r_1|)}{4\pi |r_1 - r_2|}
$$

The uncertainty in the x-component of momentum of the test charge in region 1 generates an uncertainty in $A_x$ in region 2, and therefore an uncertainty in $B_y$ and $B_z$, but not $B_x$ in 2, but it contributes nothing to the uncertainty in $E_y$ in region 2.

The uncertainty in the electric field at 2 is

$$
\Delta E(r_2, t_2) \sim -\frac{\partial}{\partial r_2} \Delta \phi(r_2, t_2) - \frac{1}{c \partial t_2} \Delta A(r_2, t_2)
$$

The uncertainty in the y-component is

$$
\Delta E_y(\Omega_2) \geq -\int_{\Omega_2} \frac{d\Omega_2}{\Omega_2} \frac{\partial}{\partial y_2} \Delta \phi(r_2, t) \\
\geq -\frac{cQ\Delta x}{4\pi V_1} \int_{\Omega_2} \frac{d\Omega_2}{\Omega_2} \int_{\Omega_1} \frac{d\Omega_1}{\Omega_1} \partial^2 \frac{\delta(r - ct)}{r}
$$

Finally

$$
\Delta E_x(\Omega_1) \Delta E_y(\Omega_2) \geq \frac{\hbar c}{8\pi} \left| \int_{\Omega_2} \frac{d\Omega_2}{\Omega_2} \int_{\Omega_1} \frac{d\Omega_1}{\Omega_1} \partial^2 \frac{\delta(r - ct)}{r} \right| \\
(2.20)
$$

same as from Equation 2.15
2.4 Casimir Effect

We established that the uncertainty principle for quantized electromagnetic fields guarantees a vacuum expectation value for the fields. In particular we found that for two space-time volumes \( \Omega_1 \) and \( \Omega_2 \), with spatial dimension \( L \), and separated by distance \( r \) and time \( t \) that

\[
\Delta E_x(\Omega_1)\Delta E_y(\Omega_2) \geq \frac{\hbar c}{8\pi} \left\lvert \int_{\Omega_2} \frac{d\Omega_2}{\Omega_2} \int_{\Omega_1} \frac{d\Omega_1}{\Omega_1} \frac{\partial^2}{\partial x_1 \partial y_2} \frac{\delta(r - ct)}{r}\right\rvert \sim \frac{\hbar c}{8\pi L^4}
\]

We assume that \( r \sim L \), that is, the two volumes are adjacent. Then the fluctuation of the field

\[
\Delta E \sim \Delta B \sim \sqrt{\frac{\hbar c}{L}}.
\]

The fluctuation in the energy

\[
\Delta H \sim (\Delta E)^2 dV \sim \frac{\hbar c}{L^4} L^3 \sim \frac{\hbar c}{L}
\]

This is the energy of a single photon of wavelength \( L \). The fluctuation in the field strengths are due to changes in occupation number of order 1 for a photon with energy \( \hbar c/L \). The vacuum energy is infinite, but the fluctuations corresponds to the longest wavelength photon that can fit in the volume. The larger the volume, the smaller the fluctuation.

Suppose we have a pair of parallel conducting plates in vacuum, perpendicular to the z-axis, with length and width \( L \), and separation \( z = d \). The standing wave electric field between the plates that satisfies the boundary conditions is

\[
\psi_n(x, y, z, t) = e^{-i\omega_n t} e^{i(k_x x + k_y y)} \sin(k_n z)
\]

where \( k_n = \sqrt{k_x^2 + k_y^2 + (\frac{n\pi}{z})^2} \). The expectation value of the square of the electric field in the vacuum state is

\[
\langle E^2 \rangle = \langle 0 \mid E^2 \mid 0 \rangle = \sum_k \frac{\hbar c k}{2V} \langle 0 \mid a_k a_k^\dagger \mid 0 \rangle.
\]

The zero point energy, vacuum energy, ground state energy is \( \sum_i \frac{1}{2} h\omega_i \). The total energy between the plates is

\[
E(z) = \frac{1}{2} \sum_{n_x} \sum_{n_y} \sum_{n_z} \frac{\hbar c}{L} \sqrt{\left(\frac{n_x \pi}{L}\right)^2 + \left(\frac{n_y \pi}{L}\right)^2 + \left(\frac{n_z \pi}{z}\right)^2}
\]

\[
\rightarrow \frac{1}{2} \frac{\hbar c}{\pi} \int\frac{L dk_x}{\pi} \int\frac{L dk_y}{\pi} \sum_{n_z=0}^\infty \epsilon_n \sqrt{k_x^2 + k_y^2 + \left(\frac{n_z \pi}{z}\right)^2}
\]

\( \epsilon_0 = 1 \) and \( \epsilon_n > 0 = 1 \) as there are two polarizations for all but the \( n = 0 \) mode. The sums in \( x \) and \( y \) propagation directions become continuous for \( L \) very large. The sums /integrals clearly diverge so we introduce a cutoff. In the end we are interested in the difference of the energy with and without the plates and in that difference the cutoff will drop out. Meanwhile define the cutoff function \( f(k_n a) \) with the property that for \( k_n a \ll 1 \), \( f \rightarrow 1 \), and for \( ka \gg 1 \), \( f(ka) \rightarrow 0 \).
2.4. CASIMIR EFFECT

Then
\[ E(z) \rightarrow \frac{1}{2} \hbar c \int \frac{Ldk_n}{\pi} \int \frac{Ldk_n}{\pi} \sum_{n_z=0}^{\infty} \epsilon_n \sqrt{k_z^2 + k_y^2 + \left(\frac{n_z \pi}{z}\right)^2} f(ak_n(z)) \] (2.21)

where
\[ k_n(z) = \sqrt{k_z^2 + k_y^2 + \left(\frac{n_z \pi}{z}\right)^2}. \]

If \( z = D \sim L \), that is if the separation of the plates is so large that we can treat the wave numbers as continuous, Equation 2.21 would become
\[ E_\infty(D) = \hbar c \frac{L^2}{4} \int_0^\infty \kappa^2 d\kappa \int_0^\infty \frac{Ddk_n}{\pi} \sqrt{k_z^2 + \kappa^2} f(a\sqrt{k_z^2 + \kappa^2}) \] (2.22)

In that last the integrations
\[ \int_0^\infty dk_x \int_0^\infty dk_y = \frac{1}{4} \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y = \frac{2\pi}{4} \int_0^\infty \kappa^2 d\kappa. \]

Also when continuous the fact that there is only a single \( n_z = 0 \) mode is irrelevant. Finally, the different in the total energy due to the plates is
\[ \delta E = E(z) - E_\infty(z) \]

And
\[ \delta E(z) = \hbar c \frac{L^2}{4\pi} \int_0^\infty dk^2 \left\{ \sum_n \epsilon_n \sqrt{n^2 + \frac{u f((a\pi/z)\sqrt{n^2 + u}) - z^2}{\pi} \int_0^\infty dk_z \sqrt{k_z^2 + \kappa^2} f(a\sqrt{k_z^2 + \kappa^2}) \right\} \] (2.23)

In anticipation of dealing with Equation 2.23 define \( u = (zk_z/\pi)^2 \) and we have
\[ \delta E(z) = \hbar c \frac{L^2}{4\pi} \left(\frac{\pi}{z}\right)^3 \int_0^\infty du \left\{ \sum_n \epsilon_n \sqrt{n^2 + u} f((a\pi/z)\sqrt{n^2 + u}) - \int_0^\infty dn \sqrt{n^2 + u} f((a\pi/z)\sqrt{n^2 + u}) \right\} \] (2.24)

In order to evaluate Equation 2.24 we use the Euler-Maclaurin formula which connects integrals with sums.
\[ \sum_{i=0}^\infty f(i) = \int_0^\infty f(x)dx - B_1(f(\infty) + f(0)) + \sum_{r=1}^{\infty} \frac{B_{2r}}{(2r)!} f^{(2r-1)}(\infty) - f^{(2r-1)}(0) + R \]

where \( B_j \) are Bernoulli numbers, (which are zero for all odd \( j \) except \( j = 1 \), and \( R \) is a remainder that is small, and \( f^j = d^j f(x)/dx^j \). If we define
\[ F(n) = \int_{n^2}^{\infty} w^{\frac{1}{2}} f(w^{\frac{1}{2}})dw \]

then
\[ \delta E(z) = \hbar c \frac{L^2}{4\pi} \left(\frac{\pi}{z}\right)^3 \left\{ \sum_n F(n) - \int_0^\infty dn F(n) \right\} \]
2.4. CASIMIR EFFECT

\[ \frac{\hbar c L^2}{4\pi} \left( \frac{\pi}{z} \right)^3 \left\{ -B_1 (F(\infty) + F(0)) + \sum_{r=1}^{\infty} \frac{B_{2r}}{(2r)!} (F^{(2r-1)}(\infty) - F^{(2r-1)}(0)) \right\} \]

\[ \sim \frac{\hbar c L^2}{4\pi} \left( \frac{\pi}{z} \right)^3 \left\{ -B_1 F(0) + \frac{B_2}{(2)!} (-F^{(3)}(0)) \right\} \]

\[ = \frac{\hbar c L^2}{4\pi} \left( \frac{\pi}{z} \right)^3 \left\{ \sum_n \epsilon(n) F(n) - \int_0^{\infty} dn F(n) \right\} = \frac{\hbar c L^2}{4\pi} \left( \frac{\pi}{z} \right)^3 \left\{ \frac{B_2}{(2)!} (-F^{(3)}(0)) \right\} \]

\[ \approx \frac{\hbar c L^2}{4\pi} \left( \frac{\pi}{z} \right)^3 \left\{ \sum_n \epsilon(n) F(n) - \int_0^{\infty} dn F(n) \right\} = \frac{\hbar c L^2}{4\pi} \left( \frac{\pi}{z} \right)^3 \left\{ \frac{B_2}{(2)!} (-F^{(3)}(0)) \right\} \]

where \( B_1 = -1/2, B_2 = 1/30, \) and \( F^3(0) = -4. \) We keep only the first term in the series and assume that \( F(\infty) = 0 \) and its derivatives.

The difference in energy per unit area of the plates is

\[ \frac{\delta E(z)}{L^2} = -\hbar c \frac{\pi^2}{z^4} \frac{1}{4!30} \]

and the pressure squeezing the plates together is (energy is gained as the plates are separated)

\[ P = -\hbar c \frac{\pi^2}{z^4} \frac{3}{4!30} = -\hbar c \frac{\pi^2}{240z^4} \]