March 13, 2015
Lecture XX

Quantization of the E-M field

2.0.1 Parity and Time Reversal

On space inversion, $\mathbf{E}(\mathbf{r},t) \to -\mathbf{E}(-\mathbf{r},t)$. Think about a pair of capacitor plates at $z = \pm a$. Reflection in the x-y plane reverses $\mathbf{E}$. Then since $\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$, it must be that under inversion $\mathbf{A}(\mathbf{r},t) \to -\mathbf{A}(-\mathbf{r},t)$. And since $\mathbf{B} = \nabla \times \mathbf{A}$, under inversion $\mathbf{B}(\mathbf{r},t) \to \mathbf{B}(-\mathbf{r},t)$. To determine how the creation and annihilation operator transform consider

$$\begin{aligned}
\mathbf{A}(\mathbf{r},t) &= \sqrt{\hbar c}\int \frac{d^3\mathbf{k}}{\sqrt{(2\pi)^3 2k}}\sum_\lambda [e^{i(\mathbf{k}\cdot\mathbf{r})}\mathbf{e}_\lambda(\mathbf{k})a_\lambda(\mathbf{k},t) + h.c.] && (2.1)\\
P^\dagger \mathbf{A}(\mathbf{r},t)P &= \sqrt{\hbar c}\int \frac{d^3\mathbf{k}}{\sqrt{(2\pi)^3 2k}}\sum_\lambda [e^{i(\mathbf{k}\cdot\mathbf{r})}\mathbf{e}_\lambda(\mathbf{k})P^\dagger a_\lambda(\mathbf{k},t)P + h.c.] && (2.2)\\
-\mathbf{A}(-\mathbf{r},t) &= -\sqrt{\hbar c}\int \frac{d^3\mathbf{k}}{\sqrt{(2\pi)^3 2k}}\sum_\lambda [e^{i(-\mathbf{k}\cdot\mathbf{r})}\mathbf{e}_\lambda(\mathbf{k})a_\lambda(\mathbf{k},t) + h.c.] && (2.3)\\
& && (2.4)
\end{aligned}$$

Then in the last $\mathbf{k} \to -\mathbf{k}$,

$$\begin{aligned}
-\mathbf{A}(-\mathbf{r},t) &= -\sqrt{\hbar c}\int \frac{d^3\mathbf{k}}{\sqrt{(2\pi)^3 2k}}\sum_\lambda [e^{i(\mathbf{k}\cdot\mathbf{r})}\mathbf{e}_\lambda(-\mathbf{k})a_\lambda(-\mathbf{k},t) + h.c.] && (2.5)\\
&= -\sqrt{\hbar c}\int \frac{d^3\mathbf{k}}{\sqrt{(2\pi)^3 2k}}\sum_\lambda [e^{i(\mathbf{k}\cdot\mathbf{r})}\mathbf{e}_{-\lambda}(\mathbf{k})a_\lambda(-\mathbf{k},t) + h.c.] && (2.6)\\
&= -\sqrt{\hbar c}\int \frac{d^3\mathbf{k}}{\sqrt{(2\pi)^3 2k}}\sum_\lambda [e^{i(\mathbf{k}\cdot\mathbf{r})}\mathbf{e}_\lambda(\mathbf{k})a_{-\lambda}(-\mathbf{k},t) + h.c.] && (2.7)\\
& && (2.8)
\end{aligned}$$

Comparing Equations 2.2 and 2.7 we find that

$$P^\dagger a_\lambda(\mathbf{k},t)P = -a_{-\lambda}(-\mathbf{k},\mathbf{t})$$

Evidently the intrinsic parity of a photon is odd. Like the electric field. Think about the emission of a photon by an atom that transitions from l to $l \pm 1$, changing the sign of its parity.

Similarly

$$\begin{aligned}
\mathbf{A}(\mathbf{r},t) &= \sqrt{\hbar c}\int \frac{d\mathbf{k}}{\sqrt{(2\pi)^3 2k}}\sum_\lambda [e^{i(\mathbf{k}\cdot\mathbf{r})}\mathbf{e}_\lambda(\mathbf{k})a_\lambda(\mathbf{k},t) + h.c.] && (2.9)\\
I_t\mathbf{A}(\mathbf{r},t)I_t^{-1} &= -\mathbf{A}(\mathbf{r},-t) = -\sqrt{\hbar c}\int \frac{d\mathbf{k}}{\sqrt{(2\pi)^3 2k}}\sum_\lambda [e^{i(\mathbf{k}\cdot\mathbf{r})}\mathbf{e}_\lambda(\mathbf{k})a_\lambda(\mathbf{k},-t) + h.c.] && (2.10)
\end{aligned}$$

$$= \sqrt{\hbar c}\int\frac{d\mathbf{k}}{\sqrt{(2\pi)^3 2k}}\sum_\lambda[e^{-i(\mathbf{k}\cdot\mathbf{r})}\mathbf{e}^*_\lambda(\mathbf{k})I_t a_\lambda(\mathbf{k},t)I_t^{-1}+h.c.] \quad (2.11)$$

$$= -\sqrt{\hbar c}\int\frac{d\mathbf{k}}{\sqrt{(2\pi)^3 2k}}\sum_\lambda[e^{-i(\mathbf{k}\cdot\mathbf{r})}\mathbf{e}_{-\lambda}(\mathbf{k})I_t a_\lambda(\mathbf{k},t)I_t^{-1}+h.c.] \quad (2.12)$$

$$= -\sqrt{\hbar c}\int\frac{d\mathbf{k}}{\sqrt{(2\pi)^3 2k}}\sum_\lambda[e^{i(\mathbf{k}\cdot\mathbf{r})}\mathbf{e}_{\lambda}(\mathbf{k})I_t a_\lambda(-\mathbf{k},t)I_t^{-1}+h.c.] \quad (2.13)$$

Then Equation 2.10 is the same as Equation 2.13 if $I_t a_\lambda(\mathbf{k},t)I_t^{-1} = a_\lambda(-\mathbf{k},-t)$. In the last two steps we use $\mathbf{e}^*_\lambda(\mathbf{k}) = -\mathbf{e}_{-\lambda}(\mathbf{k})$ and then $\mathbf{e}_\lambda(-\mathbf{k}) = \mathbf{e}_{-\lambda}(\mathbf{k})$.

Higgs spin from $\rightarrow \gamma\gamma$

Let's assume that Parity is conserved, that the initial state and final state both have definite parity. A single photon state with definite helicity $a^\dagger_\lambda|\,0\rangle$ has odd parity. That is

$$P|\,\mathbf{k},\lambda\rangle = -|\,-\mathbf{k},-\lambda\rangle$$

It turns out that particle and anti-particle always have opposite parity. Therefore, a positronium s-state has odd parity. It can decay to two photons, and since parity is conserved, the two photon final state also has odd parity. If the final state is two photons with definite helicity, the parity is even. In general the effect of the parity operator on a two photon state is

$$\begin{aligned}
|\,\rangle &= |\,\mathbf{k},\lambda\rangle_1|\,-\mathbf{k},\lambda\rangle_2\\
P|\,\rangle &= P|\,\mathbf{k},\lambda\rangle_1|\,-\mathbf{k},\lambda\rangle_2\\
&= |\,-\mathbf{k},-\lambda\rangle_1|\,\mathbf{k},-\lambda\rangle_2
\end{aligned}$$

However we can construct a final state that is not simply a product of definite helicity photon states. Suppose for example our two photon state

$$\begin{aligned}
|\,\gamma\gamma\rangle &= \frac{1}{2}[|\,\mathbf{k},\lambda\rangle_1 - |\,\mathbf{k},-\lambda\rangle_1][|\,-\mathbf{k},\lambda\rangle_2 + |\,-\mathbf{k},-\lambda\rangle_2]\\
&= \frac{1}{2}\left(|\,\mathbf{k},\lambda\rangle_1|\,-\mathbf{k},\lambda\rangle_2 - |\,\mathbf{k},-\lambda\rangle_1|\,-\mathbf{k},\lambda\rangle_2 + |\,\mathbf{k},\lambda\rangle_1|\,-\mathbf{k},-\lambda\rangle_2 - |\,\mathbf{k},-\lambda\rangle_1|\,-\mathbf{k},-\lambda\rangle_2\right)\\
P|\,\rangle &= \frac{1}{2}[|\,-\mathbf{k},-\lambda\rangle_1 - |\,-\mathbf{k},\lambda\rangle_1][|\,\mathbf{k},-\lambda\rangle_2 + |\,\mathbf{k},\lambda\rangle_2]\\
&= \frac{1}{2}\left(|\,-\mathbf{k},-\lambda\rangle_1|\,\mathbf{k},-\lambda\rangle_2 - |\,-\mathbf{k},\lambda\rangle_1|\,\mathbf{k},-\lambda\rangle_2 + |\,-\mathbf{k},-\lambda\rangle_1|\,\mathbf{k},\lambda\rangle_2 - |\,-\mathbf{k},\lambda\rangle_1|\,\mathbf{k},\lambda\rangle_2\right)
\end{aligned}$$

Rotate 180° so that $\mathbf{k}\rightarrow-\mathbf{k}$. Then

$$\begin{aligned}
RP|\,\rangle &= \frac{1}{2}\left(|\,\mathbf{k},-\lambda\rangle_1|\,-\mathbf{k},-\lambda\rangle_2 - |\,\mathbf{k},\lambda\rangle_1|\,-\mathbf{k},-\lambda\rangle_2 + |\,\mathbf{k},-\lambda\rangle_1|\,-\mathbf{k},\lambda\rangle_2 - |\,\mathbf{k},\lambda\rangle_1|\,-\mathbf{k},\lambda\rangle_2\right)\\
&= (-1)|\,\rangle
\end{aligned}$$

The two photon state with linear combinations of helicity states corresponding to perpendicular linear polarization states has odd parity. The two photon state with parallel linear polarization states has even parity. π^0 is quark antiquark, has odd parity and decays to two photons with prependicular linear polarization. The Higgs has spin parity 0^+. So if we can measure the photon polarization we can determine the parity. But that does not quite give us the total spin. We need to assume that there is some preferred helicity in the intermediate state that decays to the Higgs to determine the spin.

2.1 Commutation Rules for Field Operators

As the field operators correspond to local observables, (defined at each space time point), the operators in general do not commute and there is an associated uncertainty in measurement of the fields. Since we know how to write the fields in terms of the creation and annihilation operators, and we know how they commute, we can develop the commutation rules of the fields themselves. Beginning with the vector potential,

$$A_i(x_1) = \sqrt{\hbar c}\int \frac{d^3\mathbf{k}}{\sqrt{2k(2\pi)^3}}\sum_\lambda [e^i_\lambda(\mathbf{k})a_\lambda(\mathbf{k})e^{ik\cdot x_1} + h.c.]$$

at two distinct space time points x_1 and x_2,

$$\begin{aligned} [A_i(x_1), A_j(x_2)] &= \hbar c\int \frac{d\mathbf{k}}{2k(2\pi)^3}[e^{ik\cdot(x_1-x_2)}t_{ij} - c.c.] && (2.14)\\ &= \hbar c\int \frac{d\mathbf{k}}{2k(2\pi)^3}[e^{i\mathbf{k}\cdot(\mathbf{r_1}-\mathbf{r_2})}e^{i|k|c(t_1-t_2)}t_{ij} - c.c.] && (2.15)\end{aligned}$$

where

$$t_{ij} = \sum_\lambda (e^i_{k\lambda})^* e^j_{k\lambda}$$

and

$$[a_\lambda(\mathbf{k}), a^\dagger_{\lambda'}(\mathbf{k})] = \delta_{\lambda\lambda'}\delta(\mathbf{k}-\mathbf{k}')$$

Now consider t_{ij}. Think about the cartesian representation of the polarization unit vectors, e_α. Because the polarization is transverse to $\mathbf{k}$, $\alpha = 1, 2$ and not 3. If the sum were $\alpha = 1, 2, 3$ then $\sum_\alpha e^i_\alpha$ would be the projection of a unit vector onto i and it would be perpendicular to the projection of that same unit vector onto j and we would have that $\sum_\alpha (e^i_\alpha)^* e^j_\alpha = \delta_{ij}$. But α only runs from 1 to 2. The 3 component is in the $\mathbf{k}$ direction. Therefore

$$t_{ij} = \delta_{ij} - \frac{k_i k_j}{k^2}$$

The commutator for the electric field is

$$[E_i(x_1), E_j(x_2)] = \frac{1}{c^2}[\frac{\partial A_i(x_1)}{\partial t_1}.\frac{\partial A_j(x_2)}{\partial t_2}] = \frac{1}{c^2}\frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2}[A_i(x_1), A_j(x_2)]$$

Since

$$\frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2}e^{i(\mathbf{k}\cdot\mathbf{r}-|k|ct)} = -k^2c^2e^{i(\mathbf{k}\cdot\mathbf{r}-|k|ct)}$$

and

$$\frac{\partial}{\partial r_{1i}}\frac{\partial}{\partial r_{2j}}e^{i(\mathbf{k}\cdot\mathbf{r}-|k|ct)} = -k_ik_je^{i(\mathbf{k}\cdot\mathbf{r}-|k|ct)}$$

we can write

$$\frac{k_1k_j}{k^2c^2}\frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2}e^{i(\mathbf{k}\cdot\mathbf{r}-|k|ct)} = \frac{\partial}{\partial r_{1i}}\frac{\partial}{\partial r_{2j}}e^{i(\mathbf{k}\cdot\mathbf{r}-|k|ct)}$$

Then

$$[E_i(x_1), E_j(x_2)] = 2i\hbar c\left(\delta_{ij}\frac{1}{c^2}\frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2} - \frac{\partial}{\partial r_{1i}}\frac{\partial}{\partial r_{2j}}\right)\mathcal{D}(r,t)$$

where

$$\mathcal{D}(r,t) = -\int \frac{d^3\mathbf{k}}{2k(2\pi)^3}[e^{i(\mathbf{k}\cdot\mathbf{r}-|k|ct)} - c.c.] = \frac{1}{8\pi r}[\delta(r+ct) - \delta(r-ct)].$$

Field operators evidently commute if localized to points in space time that cannot be connected by a light signal.

We find that

$$[B_i(x_1), B_j(x_2)] \quad = \quad [E_i(x_1), E_j(x_2)]$$

$$[E_i(x_1), B_i(x_2)] = 0$$

$$[E_i(x_1), B_j(x_2)] = -2i\hbar\epsilon_{ijk}\frac{\partial}{\partial t_1}\frac{\partial}{\partial r_{2k}}\mathcal{D}(r,t)$$

At equal times $t_1 = t_2$,

$$[E_i(r_1,t), E_j(r_2,t)] = [B_i(r_1,t, B_j(r_2,t)] = 0$$

but

$$[E_i(r_1,t), B_j(r_2,t)] \neq 0$$

so electric and magnetic fields cannot be specified simultaneously at all points in space.

2.2 Uncertainty Relations with EM fields

Uncertainty relations are related to commmutators according to

$$\Delta V_1 \Delta V_2 \geq \frac{1}{2}|\langle[V_1, V_2]\rangle|,$$

where V_1 and V_2 are operators. Suppose that we have two small regions of space-time volumes Ω_1 and Ω_2 respectively and that Ω_2 is in the future with respect to Ω_1. (T_1 and T_2 are the extent in time of each region, $T_1 = t_1' - t_1$, etc. The field averaged over the volume is

$$\mathbf{E}(\Omega) = \frac{1}{\Omega}\int_\Omega d\Omega \mathbf{E}(x)$$

The commutator for perpendicular (non-parallel) components of $\mathbf{E}$ in regions Ω_1 and Ω_2 is

$$\begin{aligned}
[E_i(x_1), E_j(x_2)] &= 2i\hbar c\left(\delta_{ij}\frac{1}{c^2}\frac{\partial^2}{\partial t_1 \partial t_2} - \frac{\partial^2}{\partial r_{1i}\partial r_{2j}}\right)D(r,t) && (2.16)\\
\rightarrow &= -2i\hbar c\left(\frac{\partial^2}{\partial r_{1i}\partial r_{2j}}\right)D(r,t) && (2.17)\\
&= -\frac{2}{8\pi}i\hbar c\frac{\partial^2}{\partial x_1 \partial y_2}\frac{\delta(r-ct)}{r} && (2.18)\\
\rightarrow \Delta E_x(\Omega_1)\Delta E_y(\Omega_2) &\geq \frac{\hbar c}{8\pi}\int_{\Omega_1}\frac{d\Omega_1}{\Omega_1}\int_{\Omega_2}\frac{d\Omega_2}{\Omega_2}\frac{\partial^2}{\partial x_1 \partial y_2}\frac{\delta(r-ct)}{r} && (2.19)
\end{aligned}$$

Very interesting. We evidently cannot determine orthogonal components of the electric field that can be connected by a light signal with arbitrary precision. The product of the uncertainties does however decrease with the spatial separation of the points. Which makes sense. We don't expect a very distant disturbance to have much of an effect locally.

The uncertainty in fields in Equation 2.19 follows from the construction of vector potential and then field operators in terms of creation and annihilation operators, and the commutator of those operators that follows from their association with canonical variables that behaved like P and Q and where hamilton's equations were equivalent to Maxwell's equations. We have not connected them in any way with uncertainty in real momentum and position space. That is the next step. We attempt to determine the electric field along the x-direction in volume Ω_1 by measuring the change in the momentum of a charge that is accelerated across the volume. Because of the fundamental limit on how well we can measure momentum, there is a limit on how well we measure the electric field. Meanwhile, the test charge induces a scalar and vector potential in volume Ω_2. There will be some uncertainty in the fields in region Ω_2 since we are not sure where precisely the test charge is located or how fast it is moving. The product of the uncertainties of the fields in the two regions is an independent check on the consistency of the quantization formalism.

First we determine the x-component of the E-field in region Ω_1 with test charge Q.

$$\begin{aligned}
E_x(\Omega_1) &\sim \frac{p_x(t_1') - p_x(t_1)}{Q(t_1' - t_1)}\\
\rightarrow \Delta E_x(\Omega_1) &\geq \frac{\hbar}{2QT_1\Delta x}
\end{aligned}$$

where the uncertainty in the change in p is related to the uncertainty in position Δx. In region Ω_2 there is a scalar and vector potential associated with Q in region Ω_1. The scalar potential due to the charge is

$$\begin{aligned}
\phi(\mathbf{r}_2, t_2) &= \int_{\Omega_1} d\mathbf{r}_1 c dt_1 \rho(\mathbf{r_1}, t) \frac{\delta(c(t_2 - t_1) - |\mathbf{r_2} - \mathbf{r_1}|)}{4\pi|\mathbf{r_1} - \mathbf{r_2}|} \\
&= \frac{Q}{V_1} \int_{\Omega_1} d\mathbf{r}_1 c dt_1 \frac{\delta(c(t_2 - t_1) - |\mathbf{r_2} - \mathbf{r_1}|)}{4\pi|\mathbf{r_1} - \mathbf{r_2}|} \\
\to \Delta\phi(\mathbf{r_2}, t_2) &= \frac{cQ\Delta x}{V_1} \int_{\Omega_1} d\mathbf{r}_1 c dt_1 \frac{\partial}{\partial x_1} \frac{\delta(c(t_2 - t_1) - |\mathbf{r_2} - \mathbf{r_1}|)}{4\pi|\mathbf{r_1} - \mathbf{r_2}|}
\end{aligned}$$

The uncertainty in the x-component of momentum of the test charge in region 1 generates an uncertainty in A_x in region 2, and therefore an uncertainty in B_y and B_z, but not B_x in 2, but it contributes nothing to the uncertainty in E_y in region 2.

The uncertainty in the electric field at 2 is

$$\Delta\mathbf{E}(\mathbf{r_2}, t_2) \sim -\frac{\partial}{\partial \mathbf{r_2}} \Delta\phi(\mathbf{r_2}, t_2) - \frac{1}{c}\frac{\partial}{\partial t_2} \Delta\mathbf{A}(\mathbf{r_2}, t_2)$$

The uncertainty in the y-component is

$$\begin{aligned}
\Delta E_y(\Omega_2) &\geq -\int_{\Omega_2} \frac{d\Omega_2}{\Omega_2} \frac{\partial}{\partial y_2} \Delta\phi(\mathbf{r_2}, t) \\
&\geq -\frac{cQ\Delta x}{4\pi V_1} \int_{\Omega_2} \frac{d\Omega_2}{\Omega_2} \int_{\Omega_1} d\Omega_1 \frac{\partial^2}{\partial x_1 \partial y_2} \frac{\delta(r - ct)}{r}
\end{aligned}$$

Finally

$$\Delta E_x(\Omega_1)\Delta E_y(\Omega_2) \geq \frac{\hbar c}{8\pi} \left| \int_{\Omega_2} \frac{d\Omega_2}{\Omega_2} \int_{\Omega_1} \frac{d\Omega_1}{\Omega_1} \frac{\partial^2}{\partial x_1 \partial y_2} \frac{\delta(r - ct)}{r} \right| \tag{2.20}$$

same as from Equation 2.15