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Lecture XXII

Quantization of the E-M field

2.1 Uncertainty Relations with EM fields

Uncertainty relations are related to commmutators according to

$$\Delta V_1 \Delta V_2 \geq \frac{1}{2}|\langle[V_1, V_2]\rangle|,$$

where V_1 and V_2 are operators. Suppose that we have two small regions of space-time volumes Ω_1 and Ω_2 respectively and that Ω_2 is in the future with respect to Ω_1. (T_1 and T_2 are the extent in time of each region, $T_1 = t_1' - t_1$, etc. The field averaged over the volume is

$$\mathbf{E}(\Omega) = \frac{1}{\Omega}\int_\Omega d\Omega \mathbf{E}(x)$$

The commutator for perpendicular (non-parallel) components of $\mathbf{E}$ in regions Ω_1 and Ω_2 is

$$\begin{aligned} [E_i(x_1), E_j(x_2)] &= 2i\hbar c\left(\delta_{ij}\frac{1}{c^2}\frac{\partial^2}{\partial t_1 \partial t_2} - \frac{\partial^2}{\partial r_{1i}\partial r_{2j}}\right)D(r,t) && (2.1)\\ \rightarrow &= -2i\hbar c\left(\frac{\partial^2}{\partial r_{1i}\partial r_{2j}}\right)D(r,t) && (2.2)\\ &= -\frac{2}{8\pi}i\hbar c\frac{\partial^2}{\partial x_1 \partial y_2}\frac{\delta(r-ct)}{r} && (2.3)\\ \rightarrow \Delta E_x(\Omega_1)\Delta E_y(\Omega_2) &\geq \frac{\hbar c}{8\pi}\int_{\Omega_1}\frac{d\Omega_1}{\Omega_1}\int_{\Omega_2}\frac{d\Omega_2}{\Omega_2}\frac{\partial^2}{\partial x_1 \partial y_2}\frac{\delta(r-ct)}{r} && (2.4)\end{aligned}$$

Very interesting. We evidently cannot determine orthogonal components of the electric field that can be connected by a light signal with arbitrary precision. The product of the uncertainties does however decrease with the spatial separation of the points. Which makes sense. We don't expect a very distant disturbance to have much of an effect locally.

The uncertainty in fields in Equation 2.4 follows from the construction of vector potential and then field operators in terms of creation and annihilation operators, and the commutator of those operators that follows from their association with canonical variables that behaved like P and Q and where hamilton's equations were equivalent to Maxwell's equations. We have not connected them in any way with uncertainty in real momentum and position space. That is the next step. We attempt to determine the electric field along the x-direction in volume Ω_1 by measuring the change in the momentum of a charge that is accelerated across the volume. Because of the fundamental limit on how well we can measure momentum, there is a limit on how well we measure the electric field. Meanwhile, the test charge induces a scalar and vector potential in volume Ω_2. There will be some uncertainty in the fields in region Ω_2 since we are not sure where precisely the test charge is located or how fast it is moving. The product of the uncertainties of the fields in the two regions is an independent check on the consistency of the quantization formalism.

First we determine the x-component of the E-field in region Ω_1 with test charge Q.

$$\begin{aligned} E_x(\Omega_1) &\sim \frac{p_x(t_1') - p_x(t_1)}{Q(t_1' - t_1)} \\ \rightarrow \Delta E_x(\Omega_1) &\geq \frac{\hbar}{2QT_1\Delta x} \end{aligned}$$

where the uncertainty in the change in p is related to the uncertainty in position Δx. In region Ω_2 there is a scalar and vector potential associated with Q in region Ω_1. The scalar potential due to the charge is

$$\begin{aligned} \phi(\mathbf{r}_2, t_2) &= \int_{\Omega_1} d\mathbf{r}_1 c dt_1 \rho(\mathbf{r_1}, t) \frac{\delta(c(t_2 - t_1) - |\mathbf{r_2} - \mathbf{r_1}|)}{4\pi|\mathbf{r_1} - \mathbf{r_2}|} \\ &= \frac{Q}{V_1} \int_{\Omega_1} d\mathbf{r}_1 c dt_1 \frac{\delta(c(t_2 - t_1) - |\mathbf{r_2} - \mathbf{r_1}|)}{4\pi|\mathbf{r_1} - \mathbf{r_2}|} \\ \rightarrow \Delta\phi(\mathbf{r_2}, t_2) &= \frac{cQ\Delta x}{V_1} \int_{\Omega_1} d\mathbf{r}_1 c dt_1 \frac{\partial}{\partial x_1} \frac{\delta(c(t_2 - t_1) - |\mathbf{r_2} - \mathbf{r_1}|)}{4\pi|\mathbf{r_1} - \mathbf{r_2}|} \end{aligned}$$

The uncertainty in the x-component of momentum of the test charge in region 1 generates an uncertainty in A_x in region 2, and therefore an uncertainty in B_y and B_z, but not B_x in 2, but it contributes nothing to the uncertainty in E_y in region 2.

The uncertainty in the electric field at 2 is

$$\Delta\mathbf{E}(\mathbf{r_2}, t_2) \sim -\frac{\partial}{\partial \mathbf{r_2}} \Delta\phi(\mathbf{r_2}, t_2) - \frac{1}{c}\frac{\partial}{\partial t_2} \Delta\mathbf{A}(\mathbf{r_2}, t_2)$$

The uncertainty in the y-component is

$$\begin{aligned} \Delta E_y(\Omega_2) &\geq -\int_{\Omega_2} \frac{d\Omega_2}{\Omega_2} \frac{\partial}{\partial y_2} \Delta\phi(\mathbf{r_2}, t) \\ &\geq -\frac{cQ\Delta x}{4\pi V_1} \int_{\Omega_2} \frac{d\Omega_2}{\Omega_2} \int_{\Omega_1} d\Omega_1 \frac{\partial^2}{\partial x_1 \partial y_2} \frac{\delta(r - ct)}{r} \end{aligned}$$

Finally

$$\Delta E_x(\Omega_1)\Delta E_y(\Omega_2) \geq \frac{\hbar c}{8\pi} \left| \int_{\Omega_2} \frac{d\Omega_2}{\Omega_2} \int_{\Omega_1} \frac{d\Omega_1}{\Omega_1} \frac{\partial^2}{\partial x_1 \partial y_2} \frac{\delta(r - ct)}{r} \right| \tag{2.5}$$

same as from Equation **??**

2.2 Casimir Effect

We established that the uncertainty principle for quantized electromagnetic fields guarantees a vacuum expectation value for the fields. In particular we found that for two space-time volumes Ω_1 and Ω_2, with spatial dimension L, and separated by distance r and time t that

$$\Delta E_x(\Omega_1)\Delta E_y(\Omega_2) \geq \frac{\hbar c}{8\pi}\left|\int_{\Omega_2}\frac{d\Omega_2}{\Omega_2}\int_{\Omega_1}\frac{d\Omega_1}{\Omega_1}\frac{\partial^2}{\partial x_1 \partial y_2}\frac{\delta(r-ct)}{r}\right| \sim \frac{\hbar c}{8\pi}\frac{1}{L^4}$$

We assume that $r \sim L$, that is, the two volumes are adjacent. Then the fluctuation of the field

$$\Delta E \sim \Delta B \sim \sqrt{\hbar c}/L^2.$$

The fluctuation in the energy

$$\Delta H \sim (\Delta E)^2 dV \sim \frac{\hbar c}{L^4}L^3 \sim \frac{\hbar c}{L}$$

This is the energy of a single photon of wavelength L. The fluctuation in the field strengths are due to changes in occupation number of order 1 for a photon with energy $\hbar c/L$. The vacuum energy is infinite, but the fluctuations corresponds to the longest wavelength photon that can fit in the volume. The larger the volume, the smaller the fluctuation.

Suppose we have a pair of parallel conducting plates in vacuum, perpendicular to the z-axis, with length and width L, and separation $z = d$. The standing wave electric field between the plates that satisfies the boundary conditions is

$$\psi_n(x,y,z,t) = e^{-i\omega_n t}e^{i(k_x x + k_y y)}\sin(k_n z)$$

where $k_n = \sqrt{k_x^2 + k_y^2 + \left(\frac{n\pi}{d}\right)^2}$. The expectation value of the square of the electric field in the vacuum state is

$$\langle \mathbf{E^2} \rangle = \left\langle 0 \mid \mathbf{E^2} \mid 0 \right\rangle = \sum_k \frac{\hbar c k}{2V}\left\langle 0 \mid a_k a_k^\dagger \mid 0 \right\rangle.$$

The zero point energy, vacuum energy, ground state energy is $\sum_i \frac{1}{2}\hbar\omega_i$. The total energy between the plates is

$$\begin{aligned} E(a) &= \frac{1}{2}\sum_{n_x}\sum_{n_y}\sum_{n_z}\hbar c\sqrt{\left(\frac{n_x\pi}{L}\right)^2 + \left(\frac{n_y\pi}{L}\right)^2 + \left(\frac{n_z\pi}{z}\right)^2} \\ &\to \frac{1}{2}\hbar c\int\frac{L dk_x}{\pi}\int\frac{L dk_y}{\pi}\sum_{n_z=0}^{\infty}\epsilon_n\sqrt{k_x^2 + k_y^2 + \left(\frac{n_z\pi}{d}\right)^2} \end{aligned}$$

$\epsilon_0 = \frac{1}{2}$ and $\epsilon_{n>0} = 1$ as there are two polarizations for all but the $n = 0$ mode. The sums in x and y propagation directions become continuous for L very large. The sums /integrals clearly diverge so we introduce a cutoff. In the end we are interested in the difference of the energy with and without the plates. That difference will appear and the cutoff will drop out. Meanwhile define the cutoff function $f(k_n a)$ with the property that for $k_n a \ll 1$, $f \to 1$, and for $ka \gg 1$, $f(ka) \to 0$.

Then

$$E(d) \to \frac{1}{2}\hbar c \int \frac{Ldk_x}{\pi} \int \frac{Ldk_y}{\pi} \sum_{n_z=0}^{\infty} \epsilon_n \sqrt{k_x^2 + k_y^2 + \left(\frac{n_z\pi}{d}\right)^2} f(ak_n(d)) \tag{2.6}$$

where

$$k_n(d) = \sqrt{k_x^2 + k_y^2 + \left(\frac{n_z\pi}{d}\right)^2}.$$

To get the total energy that would be in the region bounded by the plates, if the plates were not, we would treat k_z as if it were continuous like k_x and k_y and Equation 2.6 would become

$$E_\infty(d) = \hbar c \frac{L^2}{\pi^2}\frac{\pi}{4}\int_0^\infty d\kappa^2 \int_0^\infty \frac{(d)dk_z}{\pi} \sqrt{k_x^2 + \kappa^2} f(a\sqrt{k_z^2 + \kappa^2}) \tag{2.7}$$

In that last define $\kappa^2 = k_x^2 + k_y^2$ and we get

$$\int_0^\infty dk_x \int_0^\infty = \frac{1}{4}\int_{-\infty}^\infty dk_x \int_{-\infty}^\infty dk_y = \frac{2\pi}{4}\int_0^\infty \kappa^2 d\kappa.$$

Also when continuous the fact that there is only a single $n_z = 0$ mode is irrelevant. The difference in the total energy in the region of width d with and without the plates is

$$\delta E = E(d) - E_\infty(d)$$

And

$$\begin{aligned} \delta E(d) \quad &= \quad \hbar c \frac{L^2}{4\pi}\int_0^\infty d\kappa^2 \left\{ \sum_n \epsilon_n \sqrt{(n\pi/d)^2 + \kappa^2} f(a\sqrt{(n\pi/d)^2 + \kappa^2}) \right. \\ &\qquad \left. -\frac{d}{\pi}\int_0^\infty dk_z \sqrt{k_z^2 + \kappa^2} f(a\sqrt{k_z^2 + \kappa^2}) \right\} \end{aligned} \tag{2.8}$$

$$\tag{2.9}$$

Define $u = (\kappa d/\pi)^2$ and we have

$$\begin{aligned} E(d) \quad &= \quad \hbar c \frac{L^2}{4\pi}\left(\frac{\pi}{d}\right)^2 \int_0^\infty du \left\{ \sum_n \epsilon_n (\frac{\pi}{d}) \sqrt{n^2 + u} f((a\pi/d)\sqrt{n^2 + u}) - \right. \\ &\qquad \left. \frac{d}{\pi}\int_0^\infty dk_z \left(\frac{\pi}{d}\right) \sqrt{(k_z d/\pi)^2 + u} f((a\pi/d)\sqrt{(k_z d/\pi)^2 + u}) \right\} \end{aligned}$$

In order to make the sum over n and the integral over k_z look more alike define $n = k_z d/\pi$ and rewrite the integral so that

$$\frac{d}{\pi}\int_0^\infty dk_z \sqrt{(k_z d/\pi)^2 + \kappa^2} f(a\pi/d\sqrt{(k_z d/\pi)^2 + u}) \quad = \quad \frac{d}{\pi}(\frac{\pi}{d})\int_0^\infty dn (\frac{\pi}{d})^2 \sqrt{n^2 + u} f(a\pi/d\sqrt{n^2 + u})$$

and finally

$$\delta E(d) = \hbar c \frac{L^2}{4\pi}\left(\frac{\pi}{d}\right)^3 \int_0^\infty du \left\{ \sum_n \epsilon_n \sqrt{n^2 + u} f((a\pi/d)\sqrt{n^2 + u}) - \int_0^\infty dn \sqrt{n^2 + u} f((a\pi/d)\sqrt{n^2 + u}) \right\} \tag{2.10}$$

In order to evaluate Equation 2.10 we use the Euler-Maclaurin formula which relates integrals with sums.

$$\sum_{i=1}^{\infty} F(i) = \int_0^{\infty} F(x)dx + B_1(F(\infty) - F(0)) + \sum_{r=1}^{\infty} \frac{B_{2r}}{(2r)!}\left(F^{(2r-1)}(\infty) - F^{2r-1)}(0)\right) + R$$

where B_j are Bernoulli numbers, (which are zero for all odd j except $j = 1$, and R is a remainder that is small, and $F^j = d^j F(x)/dx^j$. If $w = n^2 + u$ and we define

$$F(n) = \int_{n^2}^{\infty} w^{\frac{1}{2}} f(w^{\frac{1}{2}}\pi a/d)dw$$

where instead of integrating u from $0 \to \infty$ we are integrating w from $n^2 \to \infty$

$$\begin{aligned}
\delta E(d) &\sim \hbar c\frac{L^2}{4\pi}\left(\frac{\pi}{d}\right)^3 \left\{\sum_{n=0}^{\infty} F(n) - \int_0^{\infty} dnF(n)\right\} \\
&= \hbar c\frac{L^2}{4\pi}\left(\frac{\pi}{d}\right)^3 \left\{\frac{1}{2}F(0) - B_1(F(\infty) - F(0)) - \sum_{r=1}^{\infty} \frac{B_{2r}}{(2r)!}(F^{(2r-1)}(\infty) - F^{(2r-1)}(0))\right\}
\end{aligned}$$

Note that the Maclaurin formula sums from 1 to ∞ but our sum starts at $n = 0$. That's why we add the extra $F(0)$ in the last formula. Now remember that the $n = 0$ term had only one polarization and all modes with $n > 1$ had two. We assume that $F(\infty)$ and deriviatives of F at infinity are all zero (thanks to the cutoff). Then

$$\delta E(d) \sim \hbar c\frac{L^2}{4\pi}\left(\frac{\pi}{d}\right)^3 \left\{\frac{1}{2}F(0) + B_1 F(0) - \frac{B_4}{(4)!}(-F^{(3)}(0))\right\}$$

Note that

$$\frac{dF(n)}{dn} = \frac{dF(w)}{dw}\frac{dw}{dn}|_{w=\infty} - \frac{dF(w)}{dw}\frac{dw}{dn}|_{w=n^2},$$

and

$$\frac{dF}{dw} = w^{\frac{1}{2}} f(w^{\frac{1}{2}}\pi a/d)$$

So

$$\frac{dF}{dn} = -nf(n\pi a/d)(2n) = -2n^2 f(n\pi a/d)$$

Then $\frac{dF(0)}{dn} = 0$, $\frac{d^3F(0)}{dn^3} = -4$. Now

$$\begin{aligned}
\delta E(d) &= \hbar c\frac{L^2}{4\pi}\left(\frac{\pi}{d}\right)^3 \left\{\sum_{n=0}^{\infty} \epsilon(n)F(n) - \int_0^{\infty} dnF(n)\right\} = \hbar c\frac{L^2}{4\pi}\left(\frac{\pi}{d}\right)^3 \left\{\frac{B_4}{(4)!}(F^{(3)}(0))\right\} \\
&= \hbar c\frac{L^2}{4\pi}\left(\frac{\pi}{d}\right)^3 \frac{-4}{4!30}
\end{aligned}$$

where $B_1 = -1/2$ and $B_4 = 1/30$. We keep only the first term in the series and assume that $F(\infty) = 0$ and its derivatives.

The difference in energy per unit area of the plates is

$$\frac{\delta E(d)}{L^2} = -\hbar c \frac{\pi^2}{d^3} \frac{1}{4!30}$$

and the pressure squeezing the plates together is (energy is gained as the plates are separated)

$$P = -\hbar c \frac{\pi^2}{d^4} \frac{3}{4!30} = -\hbar c \frac{\pi^2}{240 d^4}$$