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Lecture XXIII

Quantization of the E-M field

2.1 Casimir Effect

We established that the uncertainty principle for quantized electromagnetic fields guarantees a vacuum expectation value for the fields. In particular we found that for two space-time volumes Ω_1 and Ω_2, with spatial dimension L, and separated by distance r and time t that

$$\Delta E_x(\Omega_1)\Delta E_y(\Omega_2) \geq \frac{\hbar c}{8\pi}\left|\int_{\Omega_2}\frac{d\Omega_2}{\Omega_2}\int_{\Omega_1}\frac{d\Omega_1}{\Omega_1}\frac{\partial^2}{\partial x_1 \partial y_2}\frac{\delta(r-ct)}{r}\right| \sim \frac{\hbar c}{8\pi}\frac{1}{L^4}$$

We assume that $r \sim L$, that is, the two volumes are adjacent. Then the fluctuation of the field

$$\Delta E \sim \Delta B \sim \sqrt{\hbar c}/L^2.$$

The fluctuation in the energy

$$\Delta H \sim (\Delta E)^2 dV \sim \frac{\hbar c}{L^4}L^3 \sim \frac{\hbar c}{L}$$

This is the energy of a single photon of wavelength L. The fluctuation in the field strengths are due to changes in occupation number of order 1 for a photon with energy $\hbar c/L$. The vacuum energy is infinite, but the fluctuations corresponds to the longest wavelength photon that can fit in the volume. The larger the volume, the smaller the fluctuation.

Suppose we have a pair of parallel conducting plates in vacuum, perpendicular to the z-axis, with length and width L, and separation $z = d$. The standing wave electric field between the plates that satisfies the boundary conditions is

$$\psi_n(x,y,z,t) = e^{-i\omega_n t}e^{i(k_x x + k_y y)}\sin(k_n z)$$

where $k_n = \sqrt{k_x^2 + k_y^2 + \left(\frac{n\pi}{d}\right)^2}$. The expectation value of the square of the electric field in the vacuum state is

$$\langle \mathbf{E^2}\rangle = \langle 0 \mid \mathbf{E^2} \mid 0\rangle = \sum_k \frac{\hbar c k}{2V}\left\langle 0 \mid a_k a_k^\dagger \mid 0\right\rangle.$$

The zero point energy, vacuum energy, ground state energy is $\sum_i \frac{1}{2}\hbar\omega_i$. The total energy between the plates is

$$\begin{aligned} E(a) &= \frac{1}{2}\sum_{n_x}\sum_{n_y}\sum_{n_z}\hbar c\sqrt{\left(\frac{n_x\pi}{L}\right)^2 + \left(\frac{n_y\pi}{L}\right)^2 + \left(\frac{n_z\pi}{z}\right)^2} \\ &\rightarrow \frac{1}{2}\hbar c\int\frac{L dk_x}{\pi}\int\frac{L dk_y}{\pi}\sum_{n_z=0}^{\infty}\epsilon_n\sqrt{k_x^2 + k_y^2 + \left(\frac{n_z\pi}{d}\right)^2} \end{aligned}$$

$\epsilon_0 = \frac{1}{2}$ and $\epsilon_{n>0} = 1$ as there are two polarizations for all but the $n = 0$ mode. The sums in x and y propagation directions become continuous for L very large. The sums /integrals clearly diverge so we introduce a cutoff. In the end we are interested in the difference of the energy with and without the plates. That difference will appear and the cutoff will drop out. Meanwhile define the cutoff function $f(k_n a)$ with the property that for $k_n a \ll 1$, $f \to 1$, and for $ka \gg 1$, $f(ka) \to 0$.

Then

$$E(d) \to \frac{1}{2}\hbar c \int \frac{Ldk_x}{\pi} \int \frac{Ldk_y}{\pi} \sum_{n_z=0}^{\infty} \epsilon_n \sqrt{k_x^2 + k_y^2 + \left(\frac{n_z \pi}{d}\right)^2} f(ak_n(d)) \tag{2.1}$$

where

$$k_n(d) = \sqrt{k_x^2 + k_y^2 + \left(\frac{n_z \pi}{d}\right)^2}.$$

To get the total energy that would be in the region bounded by the plates, if the plates were not, we would treat k_z as if it were continuous like k_x and k_y and Equation 2.1 would become

$$E_\infty(d) = \hbar c \frac{L^2}{\pi^2}\frac{\pi}{4}\int_0^\infty d\kappa^2 \int_0^\infty \frac{(d)dk_z}{\pi}\sqrt{k_x^2+\kappa^2} f(a\sqrt{k_z^2+\kappa^2}) \tag{2.2}$$

In that last define $\kappa^2 = k_x^2 + k_y^2$ and we get

$$\int_0^\infty dk_x \int_0^\infty = \frac{1}{4}\int_{-\infty}^\infty dk_x \int_{-\infty}^\infty dk_y = \frac{2\pi}{4}\int_0^\infty \kappa^2 d\kappa.$$

Also when continuous the fact that there is only a single $n_z = 0$ mode is irrelevant. The difference in the total energy in the region of width d with and without the plates is

$$\delta E = E(d) - E_\infty(d)$$

And

$$\begin{aligned}\delta E(d) &= \hbar c \frac{L^2}{4\pi}\int_0^\infty d\kappa^2 \left\{\sum_n \epsilon_n \sqrt{(n\pi/d)^2+\kappa^2} f(a\sqrt{(n\pi/d)^2+\kappa^2})\right. \\ &\quad \left. -\frac{d}{\pi}\int_0^\infty dk_z \sqrt{k_z^2+\kappa^2} f(a\sqrt{k_z^2+\kappa^2})\right\} \qquad (2.3) \\ & \qquad (2.4)\end{aligned}$$

Define $u = (\kappa d/\pi)^2$ and we have

$$\begin{aligned}E(d) &= \hbar c \frac{L^2}{4\pi}\left(\frac{\pi}{d}\right)^2 \int_0^\infty du \left\{\sum_n \epsilon_n (\frac{\pi}{d})\sqrt{n^2+u} f((a\pi/d)\sqrt{n^2+u}) - \right. \\ &\quad \left. \frac{d}{\pi}\int_0^\infty dk_z \left(\frac{\pi}{d}\right)\sqrt{(k_z d/\pi)^2+u} f((a\pi/d)\sqrt{(k_z d/\pi)^2+u})\right\}\end{aligned}$$

In order to make the sum over n and the integral over k_z look more alike define $n = k_z d/\pi$ and rewrite the integral so that

$$\frac{d}{\pi}\int_0^\infty dk_z \sqrt{(k_z d/\pi)^2+\kappa^2} f(a\pi/d\sqrt{(k_z d/\pi)^2+u}) = \frac{d}{\pi}(\frac{\pi}{d})\int_0^\infty dn (\frac{\pi}{d})^2 \sqrt{n^2+u} f(a\pi/d\sqrt{n^2+u})$$

and finally

$$\delta E(d) = \hbar c \frac{L^2}{4\pi}\left(\frac{\pi}{d}\right)^3 \int_0^\infty du \left\{ \sum_n \epsilon_n \sqrt{n^2+u} f((a\pi/d)\sqrt{n^2+u}) - \int_0^\infty dn \sqrt{n^2+u} f((a\pi/d)\sqrt{n^2+u}) \right\} \quad (2.5)$$

In order to evaluate Equation 2.5 we use the Euler-Maclaurin formula which relates integrals with sums.

$$\sum_{i=1}^{\infty} F(i) = \int_0^\infty F(x)dx + B_1(F(\infty) - F(0)) + \sum_{r=1}^{\infty} \frac{B_{2r}}{(2r)!}\left(F^{(2r-1)}(\infty) - F^{2r-1)}(0)\right) + R$$

where B_j are Bernoulli numbers, (which are zero for all odd j except $j = 1$, and R is a remainder that is small, and $F^j = d^j F(x)/dx^j$. If $w = n^2 + u$ and we define

$$F(n) = \int_{n^2}^{\infty} w^{\frac{1}{2}} f(w^{\frac{1}{2}}\pi a/d)dw$$

where instead of integrating u from $0 \to \infty$ we are integrating w from $n^2 \to \infty$

$$\begin{aligned} \delta E(d) &\sim \hbar c \frac{L^2}{4\pi}\left(\frac{\pi}{d}\right)^3 \left\{ \sum_{n=0}^{\infty} F(n) - \int_0^\infty dn F(n) \right\} \\ &= \hbar c \frac{L^2}{4\pi}\left(\frac{\pi}{d}\right)^3 \left\{ \frac{1}{2}F(0) - B_1(F(\infty) - F(0)) - \sum_{r=1}^{\infty} \frac{B_{2r}}{(2r)!}(F^{(2r-1)}(\infty) - F^{(2r-1)}(0)) \right\} \end{aligned}$$

Note that the Maclaurin formula sums from 1 to ∞ but our sum starts at $n = 0$. That's why we add the extra $F(0)$ in the last formula. Now remember that the $n = 0$ term had only one polarization and all modes with $n > 1$ had two. We assume that $F(\infty)$ and deriviatives of F at infinity are all zero (thanks to the cutoff). Then

$$\delta E(d) \sim \hbar c \frac{L^2}{4\pi}\left(\frac{\pi}{d}\right)^3 \left\{ \frac{1}{2}F(0) + B_1 F(0) - \frac{B_4}{(4)!}(-F^{(3)}(0)) \right\}$$

Note that

$$\frac{dF(n)}{dn} = \frac{dF(w)}{dw}\frac{dw}{dn}|_{w=\infty} - \frac{dF(w)}{dw}\frac{dw}{dn}|_{w=n^2},$$

and

$$\frac{dF}{dw} = w^{\frac{1}{2}} f(w^{\frac{1}{2}}\pi a/d)$$

So

$$\frac{dF}{dn} = -nf(n\pi a/d)(2n) = -2n^2 f(n\pi a/d)$$

Then $\frac{dF(0)}{dn} = 0$, $\frac{d^3F(0)}{dn^3} = -4$. Now

$$\begin{aligned} \delta E(d) &= \hbar c \frac{L^2}{4\pi}\left(\frac{\pi}{d}\right)^3 \left\{ \sum_{n=0}^{\infty} \epsilon(n)F(n) - \int_0^\infty dn F(n) \right\} = \hbar c \frac{L^2}{4\pi}\left(\frac{\pi}{d}\right)^3 \left\{ \frac{B_4}{(4)!}(F^{(3)}(0)) \right\} \\ &= \hbar c \frac{L^2}{4\pi}\left(\frac{\pi}{d}\right)^3 \frac{-4}{4!30} \end{aligned}$$

where $B_1 = -1/2$ and $B_4 = 1/30$. We keep only the first term in the series and assume that $F(\infty) = 0$ and its derivatives.

The difference in energy per unit area of the plates is

$$\frac{\delta E(d)}{L^2} = -\hbar c \frac{\pi^2}{d^3} \frac{1}{4!30}$$

and the pressure squeezing the plates together is (energy is gained as the plates are separated)

$$P = -\hbar c \frac{\pi^2}{d^4} \frac{3}{4!30} = -\hbar c \frac{\pi^2}{240 d^4}$$

2.2 Lamb Shift

Because of the vacuum fluctuations of the E and B fields, a charged particle in vacuum is subject to fluctuating electromagnetic forces that introduce an uncertainty into its position. If the particle is in a potential, (like the coulomb potential of an atom), then the potential energy will also fluctuate (as long as there is some position dependence). We want to estimate the shift in the bound state energy due to this fluctuating potential energy.

First we see how the potential energy depends on some average spread in the position. We know that

$$\delta V(\mathbf{r}) = V(|\mathbf{r}+\delta\mathbf{r}| - V(\mathbf{r}) = \delta\mathbf{r}\cdot\nabla V + \frac{1}{2}\sum_{i,j}\delta x_i\delta x_j\frac{\partial^2 V}{\partial x_i\partial x_j} + \ldots \tag{2.6}$$

The fluctuations are in random directions and uncorrelated so the average $\langle\delta(\mathbf{r})\rangle = 0$ leaving

$$\langle\delta V(r)\rangle_0 = \frac{1}{6}\langle|\delta\mathbf{r}|^2\rangle_0\nabla^2 V \tag{2.7}$$

Next suppose that the charged particle is being pushed around by the flucuating electric field. The equation of motion for the particle is

$$m\frac{d^2}{dt^2}\delta\mathbf{r}(t) = -e\mathbf{E}(\mathbf{r},t) \sim -e\mathbf{E}(t) \tag{2.8}$$

where we assume that the particle motion is non-relativstic (generally true for atoms) and that the electric field is more or less constant over the region of motion and since the velocities $v \ll c$, magnetic fields are irrelevant. For each mode we will have an amplitude of oscillation

$$\delta\mathbf{r}_{\omega\lambda} = \frac{e}{m\omega^2}\mathbf{E}_{\omega\lambda} \tag{2.9}$$

and the expectation value of the square of the amplitude will be

$$(\Delta\mathbf{r}_{\omega\lambda})^2 = \langle|\delta\mathbf{r}_{\omega\lambda}\rangle_0|^2 = \frac{e^2}{m^2\omega^4}\langle|\mathbf{E}_{\omega\lambda}|^2\rangle_0 \tag{2.10}$$

The vacuum energy of the electric field is

$$\langle H\rangle_0 = \int_V d^3r\sum_{k.\lambda}\langle|E_{k,\lambda}|^2\rangle_0 = \frac{1}{2}\sum_{k,\lambda}\hbar\omega \tag{2.11}$$

Then the efield fluctuation $\langle|E_{k,\lambda}|^2\rangle \sim \frac{\hbar\omega}{2V}$.

$$(\Delta\mathbf{r})^2 = \sum_{k,\lambda}\frac{e^2}{m^2\omega^4}\frac{\hbar\omega}{2V} = \frac{e^2\hbar}{m^2}\int_0^\infty dk\frac{4\pi k^2}{(2\pi)^3\omega^3} = 8\alpha(\lambda_C/2\pi)^2\int_0^\infty\frac{d\omega}{\omega} \tag{2.12}$$

where we use the conversion from periodic boundary conditions to continuous spectrum of wave vectors

$$\frac{1}{V}\sum_{\mathbf{k}} \leftrightarrow \int\frac{d^3k}{(2\pi)^3}$$

and where $\lambda_C = h/mc$. The integral is divergent for both low and high ω. As long as the electron is in a bound state, it is unaffected by frequencies that would change its energy by less than some fraction of the bound state energy. So we use something like the bound state energy $\bar{E}_b$ as the lower limit of integration. At very high energy the non-relativstic formulation falls apart. Furthermore, in the more precise calculation we find that there is a change in the energy(mass) of a free electron due to the vacuum fluctuations. The change in the energy of the bound state that we are exploring will subtract the effect of the fluctuations in free space. For the time being we will use the electron mass mc^2 as the high frequency cutoff. Then

$$(\Delta\mathbf{r})^2 = 8\alpha(\lambda_C/2\pi)^2\ln(mc^2/\bar{E}_b) \tag{2.13}$$

Now let's determine the shift in the atomic bound state energy due to the change in V. The coulomb potential $V = -Ze^2/4\pi r$.

$$\delta V(\mathbf{r}) = \frac{8\alpha(\lambda_C/2\pi)^2}{6}\ln\left(\frac{mc^2}{\bar{E}_b}\right)\nabla^2 V \tag{2.14}$$

Since $\nabla^2\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} = \delta(\mathbf{r})$

$$\delta V(\mathbf{r}) = \frac{4\alpha Ze^2(\lambda_C/2\pi)^2}{3}\ln\left(\frac{mc^2}{\bar{E}_b}\right)\delta(\mathbf{r}) \tag{2.15}$$

The energy shift is the expectation value of $\langle\psi_n \mid \delta V \mid \psi_n\rangle$

$$\Delta E_n = \frac{4\alpha Ze^2(\lambda_C/2\pi)^2}{3}\ln\left(\frac{mc^2}{\bar{E}_n}\right)|\psi_{n0}(0)|^2 \tag{2.16}$$

For a hydrogenic atom

$$|\psi_{n0}(0)|^2 = \frac{(Z/a_0)^3}{\pi n^3} \tag{2.17}$$

where $a_0 = 4\pi\hbar^2/me^2$

$$\begin{aligned}
\Delta E_n &= \frac{4\alpha Ze^2\hbar^2}{3(mc)^2}\frac{Z^3(me^2)^3}{(4\pi\hbar^2)^3}\frac{1}{\pi n^3}\ln\left(\frac{mc^2}{\bar{E}_n}\right) \\
&= \frac{4\alpha}{3}Z^4 4\pi\left(\frac{e^2}{4\pi\hbar c}\right)^4 mc^2\frac{1}{\pi n^3}\ln\left(\frac{mc^2}{\bar{E}_n}\right) \\
&= \frac{32}{3}\frac{\alpha^3 Z^4}{n^3}\ln\left(\frac{mc^2}{\bar{E}_n}\right)E_0
\end{aligned}$$

and

$$E_0 = \frac{1}{2}mc^2\alpha^2$$

Our estimate is within a factor of two of the right answer. which in view of our rather cavelier choice of high and low energy cutoffs, is closer than we might have expected. For hydrogen, the shift of the ground state energy is order α^3 times the binding energy $\frac{1}{2}mc^2\alpha^2$ Only $l = 0$ states are effected since only $l = 0$ states are finite at the origin. Note that if $E_b = -\frac{1}{2}mc^2\alpha^2 = -13.6$eV then $\ln(mc^2/E_b) \sim 10$. It depends weakly on the precise choice of the cutoff. (We will come back to this when we know more about transitions.)