April 8, 2015
Lecture XXVII

Quantization of the E-M field

2.1 Plank radiation law

Suppoe we have atoms that make transitions between state A and B as follows

$$A \leftrightarrow \gamma + B$$

The higher energy state A decays to state B with emission of a photon. Then state B absorbs a photon and transitions to A. If the system is in equilibrium then

$$N(B)w_{abs} = N(A)w_{emis}$$

where $N(B), N(A)$ are the numbers of atoms in states A and B respectively. $wabs$ is the probablity that an atom in the state B absorbs a photon and w_{emis} is the probablity that an atom in state A emits a photon. If the atoms are in thermal equilibrium then

$$\frac{N(B)}{N(A)} = \frac{w_{emis}}{w_{abs}} = \frac{e^{-E_B/kT}}{e^{-E_A/kT}} = e^{\hbar\omega/kT} \tag{2.1}$$

The transition probability from state A and n photons to the state B with $n+1$ photons

$$w_{emis} \propto |\left\langle B, n_\gamma + 1 \mid e^{-i\mathbf{k}\cdot x}\epsilon^\alpha \cdot \mathbf{p}a_k \mid A, n_\gamma\right\rangle|^2 = (n_\gamma + 1)|\left\langle B \mid e^{-i\mathbf{k}\cdot x}\epsilon^\alpha \cdot \mathbf{p} \mid A\right\rangle|^2$$

The probability for transition from state B with n photons to state A with $n-1$ is

$$w_{abs} \propto |\left\langle A, n_\gamma - 1 \mid e^{+i\mathbf{k}\cdot x}\epsilon^\alpha \cdot \mathbf{p}a_k^\dagger \mid A, n_\gamma\right\rangle|^2 = (n_\gamma)|\left\langle A \mid e^{+i\mathbf{k}\cdot x}\epsilon^\alpha \cdot \mathbf{p} \mid B\right\rangle|^2$$

The ratio

$$\frac{w_{emis}}{w_{abs}} = \frac{n+1}{n}$$

Together with Equation 2.1 we find that the number of photons in thermal equilibrium with wave number $\mathbf{k}$ and polarization ϵ is

$$n = \frac{1}{e^{\hbar\omega/kT} - 1}$$

That is, we imagine that there is an oscillator with $\omega = c|\mathbf{k}|$ and that it is in equilibrium at temperature T when it is in the n^{th} energy level. Now imagine a box with walls that absorb and emit photons at all wavelengths and polarizations. It is filled with oscillators at every possible frequency. Each oscillator will be at energy

$$E_n = \frac{\hbar\omega_n}{e^{\hbar\omega/kT} - 1}$$

The total number of oscillators per unit frequency is the density of states

$$dN = 2\frac{V4\pi k^2 dk}{(2\pi)^3} = 2\frac{V4\pi\omega^2 d\omega}{(2\pi)^3 c^3}$$

The number of states per unit frequency per unit volume is

$$\rho(\omega) = \frac{8\pi\omega^2}{(2\pi)^3 c^3}$$

The total energy per unit frequency per unit volume is

$$U(\omega) = \rho(\omega) E_n = \frac{8\pi\hbar\omega^3}{(2\pi)^3 c^3 (e^{\hbar\omega/kT} - 1)}$$

2.2 Scattering and Time Dependent Perturbation Theory

With time dependent perturbation theory we calculate a rate. The Golden Rule gives us the transition rate.

$$d\Gamma = \frac{2\pi}{\hbar} |\langle f \mid H_I \mid i\rangle|^2 \rho$$

where ρ is some density states over which we can integrate the energy conserving delta-function. The rate is the product of the cross section and the flux of incoming particles. If there is a single incident particle the flux is v/V. Then the differential cross section is

$$d\sigma = \frac{d\Gamma}{v/V}$$

We imagine an incident plane wave, a scattering potential and an outgoing plane wave. The perturbation is constant in time from $t = 0$ to $t = t$. The energy conserving delta function will ensure that energy in initial and final plane wave are the same. Looks just like the first Born approximation. For nonrelativistic particles, the density of final states is

2.3 Scattering photons

Now rather than absorption or emission of a single photon we investigate photon scattering. We consider scattering from a hydrogen like, single electron atom. Initial and final states are $\mid \mathbf{k}(\epsilon), i\rangle$ and $\mid \mathbf{k}'(\epsilon'), f\rangle$. The photon in the initial state with wave number $\mathbf{k}$ and polarizaiton $\hat{\epsilon}_\lambda$ is absorbed. Atomic state $\mid i\rangle$ transitions to $\mid f\rangle$ and a photon $\mathbf{k}', \hat{\epsilon}'_{\lambda'}$ is emitted. Recall the interaction

$$H_I = \frac{e}{mc}\mathbf{p}\cdot\mathbf{A} + \frac{e^2}{2mc^2}\mathbf{A}\cdot\mathbf{A}$$

The first lowest order term absorbs or emits a single photon so cannot be responsible for photon scattering at first order in the perturbation theory. The $\mathbf{A}\cdot\mathbf{A}$ term changes the photon number by 0 or 2 so it can effect the transition, at least an elastic scatter. The field operator

$$\mathbf{A}(\mathbf{x}, t) = \frac{1}{\sqrt{V}}\sum_{\mathbf{k}}\sum_{\alpha} c\sqrt{\frac{\hbar}{2\omega}}[a_{k,\alpha}(0)e^{\alpha}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} + a^\dagger_{k,\alpha}(0)e^{\alpha}e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)}]$$

Then $\mathbf{A}\cdot\mathbf{A}$ will have the combinatons of creation and annihilation operators

$$a^\dagger_{k',\alpha'}a_{k,\alpha} + a_{k,\alpha}a^\dagger_{k',\alpha'}$$

that will connect the initial and final states.

$$\begin{aligned}\langle B, \mathbf{k}' \mid H_I \mid A, \mathbf{k}\rangle &= \frac{e^2}{2mc^2}\langle A, k' \mid \mathbf{A}\cdot\mathbf{A} \mid B, \mathbf{k}\rangle \\ &= \frac{e^2}{2mc^2}\left\langle B, \mathbf{k}' \mid (a_k a^\dagger_{k'} + a^\dagger_{k'}a_k)e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} \mid A, \mathbf{k}\right\rangle \frac{c^2\hbar}{2V\sqrt{\omega\omega'}}\epsilon\cdot\epsilon' e^{-i(\omega-\omega')t}\end{aligned}$$

And in the dipole approximation ($\mathbf{k}\cdot\mathbf{x} \ll 1$.)

$$\langle B, \mathbf{k}' \mid H_I \mid A, \mathbf{k}\rangle \sim \frac{e^2}{2mc^2}2\frac{c^2\hbar}{2V\sqrt{\omega\omega'}}\epsilon\cdot\epsilon' e^{-i(\omega-\omega')t}\langle B \mid A\rangle$$

the scattering is necessarily elastic as initial and final atomic states must be identical. The transition amplitude to first order from intial to final state is

$$c_A^1(t) = \frac{1}{i\hbar}\frac{e^2}{2mc^2}\frac{c^2\hbar}{2V\sqrt{\omega\omega'}}2\delta_{A,B}\epsilon_\lambda\cdot\epsilon'_{\lambda'}\int_0^t e^{i(\hbar\omega'+E_f-\hbar\omega-E_i)t'/\hbar}dt' \tag{2.2}$$

The transition probability

$$|c_A|^2 = \frac{2\pi}{\hbar}|\langle B \mid H_I \mid A\rangle|^2 t\delta(E_B - E_A + \hbar\omega)$$

and the transition rate

$$\Gamma = \int \frac{d|c_A|^2}{dt}\rho(E)dE = \frac{2\pi}{\hbar}|\langle B \mid H_I \mid A\rangle|^2\rho(E)$$

where $\rho(E)$ is the density of final states for the scattered photon. Putting the pieces together

$$\Gamma = \frac{2\pi}{\hbar}\left(\frac{e^2}{2mc^2}\frac{c^2\hbar}{2V\sqrt{\omega\omega'}}\right)^2 4|\hat{\epsilon}\cdot\hat{\epsilon}'|^2\frac{V}{(2\pi)^3}\frac{\omega'^2}{\hbar c^3}d\Omega$$

The differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{\text{TransitionRate}}{\text{Flux}} = \frac{\text{Rate}}{c/V}$$

where the photon flux is one per unit volume at velocity c. Then

$$\frac{d\sigma}{d\Omega} = \frac{2\pi}{\hbar}\left(\frac{e^2}{2mc^2}\frac{c^2\hbar}{2\sqrt{\omega\omega'}}\right)^2 4|\hat{\epsilon}\cdot\hat{\epsilon}'|^2\frac{1}{(2\pi)^3}\frac{\omega'^2}{\hbar c^4} = \left(\frac{e^2}{4\pi mc^2}\right)^2\left(\frac{\omega'}{\omega}\right)|\hat{\epsilon}\cdot\hat{\epsilon}'|^2 = r_0^2\left(\frac{\omega'}{\omega}\right)|\hat{\epsilon}\cdot\hat{\epsilon}'|^2$$

where r_0 is the classical electron radius.

Angular distribution

It is interesting to consider the angular distribution of the scattered photons. That is the easiest to measure experimentally. Suppose the incident photon is traveling in the $+z$ direction with circular polarization $\epsilon_+ = -\frac{1}{\sqrt{2}}(\epsilon_x + i\epsilon_y)$. It scatters at the origin. The scattered photon leaves the origin with momentum $\mathbf{k}' = |\mathbf{k}'|\hat{\mathbf{z}}'$ at angle θ with respect to the z-axis in the x-z plane. The scattered photon polarization $\epsilon'_\pm = \mp\frac{1}{\sqrt{2}}(\epsilon_{x'} \pm i\epsilon_{y'})$ where $\epsilon_{x'}, \epsilon_{y'}$ are the unit vectors orthogonal to the z' axis which coincides with the scattered photon momentum. Since

$$\hat{\mathbf{x}}' = \hat{\mathbf{x}}\cos\theta + \hat{\mathbf{z}}\sin\theta, \quad \hat{\mathbf{y}}' = \hat{\mathbf{y}}$$

we can write

$$\epsilon'_\pm = \mp\frac{1}{\sqrt{2}}(\epsilon_x\cos\theta + \epsilon_z\sin\theta \pm i\epsilon_y)$$

Then

$$\hat{\epsilon}^*\cdot\hat{\epsilon}'_\pm = \mp\frac{1}{2}(\cos\theta \pm 1)$$

And the differential cross section for scattering to both polarizations is

$$\frac{d\sigma}{dd\Omega} = r_0^2\left(\frac{\omega'}{\omega}\right)\left(|\frac{1}{2}(1+\cos\theta)|^2 + |\frac{1}{2}(1-\cos\theta)|^2\right) r_0^2\left(\frac{\omega'}{\omega}\right)\left(\frac{1}{2}(1+\cos^2\theta)\right)$$

And the total cross section (elastic scattering)

$$\sigma = r_0^2\left(\frac{\omega'}{\omega}\right)\int\left(\frac{1}{2}(1+\cos^2\theta)\right)d\Omega = \frac{8}{3}\pi r_0^2$$

Second order time dependent perturbation theory

But what about the term $\mathbf{A}\cdot\mathbf{p}$ to second order. Then we can create k' and annihilate k. Check out the Feynman diagrams. There are three. Time goes from bottom of the page to the top. In the first, an electron in an initial state $\mid i\rangle$ and a photon with $\mid \mathbf{k}, \epsilon_\lambda\rangle$ end in a vertex. An intermediate (electron) state $\mid I\rangle$ emerges from that vertex. At the next vertex an electron in state $\mid f\rangle$ and photon $\mid \mathbf{k}', \epsilon'_{\lambda'}\rangle$ emerge.

In the second, the photon $\mid k', \epsilon'_{\lambda'}\rangle$ is created leaving the electron in the intermediate state and subsequently the photon $\mid k', \epsilon_\lambda\rangle$ is absorbed. The third diagram is a seagull. No intermediate state, it corresponds to the $\mathbf{A}\cdot\mathbf{A}$ term in the interaction hamiltonian. Let's revisit second order perturbation theory. After expansion of the state $\mid \psi(t)\rangle$ in terms of time dependent coefficients of the eigenkets of the unperturbed Hamiltonian, we write

$$\dot{c}_m = \sum_k \frac{1}{i\hbar}\langle m \mid H_I \mid k\rangle\, e^{i(E_m-E_k)t/\hbar} c_k(t)$$

If at $t = 0, c_k(0) = \delta_{lk}$, the first and second order solutions are

$$\begin{aligned}
\dot{c}_m &= \frac{1}{i\hbar}\langle m \mid H_I \mid l\rangle\, e^{i(E_m-E_l)t/\hbar} \\
c_m^1(t) &= \frac{1}{i\hbar}\int^t \langle m \mid H_I \mid l\rangle\, e^{i(E_m-E_l)t'/\hbar} dt' \\
\rightarrow c_m^2(t) &= \frac{1}{i\hbar}\sum_k\int^t \langle m \mid H_I \mid k\rangle\, e^{i(E_m-E_k)t''/\hbar} dt'' c_k^1(t'') \\
&= \frac{1}{(i\hbar)^2}\sum_k\int^t \langle m \mid H_I \mid k\rangle\, e^{i(E_m-E_k)t''/\hbar} dt'' \int^{t''} \langle k \mid H_I \mid l\rangle\, e^{i(E_k-E_l)t'/\hbar} dt'
\end{aligned}$$

Next assume the usual harmonic time dependence for H_I as per the vector potential field operater. Actually the operator is a sum over all frequencies. The frequency that is singled out is that which corresponds to the difference in energy of the atomic electron states. Therefore if intial and final atomic states have different energy, the relevant frequencies for the corresponding photons are also different. Let's suppose that the photon with ω is absorbed and that with ω' is emitted. Then for the diagram where the photon $\mathbf{k}$ is absorbed before $\mathbf{k}'$ is emitted,

$$c_m^{(2)}(t)[a] = \; = \frac{1}{(i\hbar)^2}\sum_k H_I^{mk'} H_I^{kl}\int^t e^{i(E_m-E_k+\hbar\omega')t''/\hbar} dt'' \int^{t''} e^{i(E_k-E_l-\hbar\omega)t'/\hbar} dt'$$

And for the other time ordering

$$c_m^{(2)}(t)[b] = \;=\; \frac{1}{(i\hbar)^2}\sum_k H_I^{mk}H_I^{kl'}\int^t e^{i(E_m-E_k-\hbar\omega)t''/\hbar}dt''\int^{t''} e^{i(E_k-E_l+\hbar\omega')t'/\hbar}dt'$$

If we insert a regularizing $-i\epsilon$ term into the exponent of the integration over dt', and integrate from $-\infty$ to t, then

$$\int^{t''} e^{i(E_k-E_l+\hbar\omega')t'/\hbar}dt' = \frac{e^{i(E_k-E_l+\hbar\omega'-i\epsilon)t''/\hbar}}{i(E_k-E_l+\hbar\omega')}$$

Combining both terms we have

$$c_m^{(2)}(t) \;=\; \frac{1}{-\hbar^2}\sum_k\left(\frac{H_I^{mk'}H_I^{kl}}{E_k-E_l-\hbar\omega}+\frac{H_I^{mk}H_I^{kl'}}{E_k-E_l+\hbar\omega'}\right)\int^t dt''e^{i(E_m-E_l+\hbar\omega'-\hbar\omega)t''/\hbar}$$

This second order amplitude can now be combined with the first order term from Equation 2.2. If we furthermore use the dipole approximation the transition rate is

$$\begin{aligned} w &= \int(|c^1+c^2|^2/t)\rho(E)dE \\ &= \frac{2\pi}{\hbar}\left(\frac{c^2\hbar}{2V\sqrt{\omega\omega'}}\right)^2\left(\frac{2^2}{mc^2}\right)^2\frac{V}{(2\pi)^3}\frac{\omega'^2}{\hbar c^3}d\Omega \\ &\quad\times\left|\delta_{ml}\epsilon_\lambda\cdot\epsilon_{\lambda'}-\frac{1}{m}\sum_k\left(\frac{(\mathbf{p}\cdot\epsilon')_{mk}(\mathbf{p}\cdot\epsilon)_{kl}}{E_k-E_l-\hbar\omega}+\frac{(\mathbf{p}\cdot\epsilon)_{mk}(\mathbf{p}\cdot\epsilon')_{kl}}{E_k-E_l-\hbar\omega'}\right)\right|^2 \end{aligned}$$

Divide the transition rate by the flux (c/V) to determine the differential cross section. We find that

$$\frac{d\sigma}{d\Omega} = r_0^2\left(\frac{\omega'}{\omega}\right)^2\left|\delta_{ml}\epsilon_\lambda\cdot\epsilon_{\lambda'}-\frac{1}{m}\sum_k\left(\frac{(\mathbf{p}\cdot\epsilon')_{mk}(\mathbf{p}\cdot\epsilon)_{kl}}{E_k-E_l-\hbar\omega}+\frac{(\mathbf{p}\cdot\epsilon)_{mk}(\mathbf{p}\cdot\epsilon')_{kl}}{E_k-E_l-\hbar\omega'}\right)\right|^2 \qquad (2.3)$$

where $r_0 = \alpha\frac{\hbar}{mc}$ is the classical electron radius. Equation 2.3 is the Kramers-Heisenberg formula.

2.3.1 Rayleigh scattering

Rayleigh scattering is the limit where initial and final atomic states are the same, and the photon energy is small compared to the energy between the atomic states. Equation 2.3 can be simplified, using the commutation relation of $\mathbf{x}$ and $\mathbf{p}$ and the completeness of the intermediate states $\mid k\rangle$ in the appropriate limit to

$$\frac{d\sigma}{d\Omega} \;=\; \left(\frac{r_0}{m\hbar}\right)^2\omega^4\left|\sum_k\frac{1}{\omega_{kl}^3}[(\mathbf{p}\cdot\epsilon')_{lk}(\mathbf{p}\cdot\epsilon)_{kl}+(\mathbf{p}\cdot\epsilon)_{lk}(\mathbf{p}\cdot\epsilon')_{kl}]\right|^2 \qquad (2.4)$$

$$\;=\; \left(\frac{r_0 m}{\hbar}\right)^2\omega^4\left|\sum_k\frac{1}{\omega_{kl}}[(\mathbf{x}\cdot\epsilon')_{lk}(\mathbf{x}\cdot\epsilon)_{kl}+(\mathbf{x}\cdot\epsilon)_{lk}(\mathbf{x}\cdot\epsilon')_{kl}]\right|^2 \qquad (2.5)$$

The cross section scales inversely as the fourth power of the wavelength.

2.3.2 Thomson scattering

Thomson scattering is the limit where the photon energy is much greater than the binding energy. We can ignore the terms in Equation 2.3 with $\hbar\omega$ or $\hbar\omega'$ in the denominator. That leaves only the seagull diagram.

$$\frac{d\sigma}{d\Omega} = r_0^2 |\epsilon \cdot \epsilon'|^2 \tag{2.6}$$

independent of wavelength.

Classical scattering

Conceptual scattering light from a charged particle involves first accelerating the charge with the E-field in the incoming light and then radiaition by the accelerating charge. In the case of Rayleigh scattering we imagine the electron bound to an atom in a harmonic oscillator potential so that

$$\ddot{\mathbf{x}} + \omega_0^2 \mathbf{x} = -\frac{e}{m}\mathbf{E}_0 e^{-i\omega t}$$

Then

$$\ddot{\mathbf{x}} = -\frac{e}{m}\frac{\omega^2}{\omega_0^2 - \omega^2} E_0 e^{-i\omega t}$$

The power radiated by an accelerating charge

$$P \sim \frac{e^2 \ddot{\mathbf{x}}^2}{c^3} \sim \frac{e^2}{c^3}\left(\frac{e}{m}\frac{\omega^2}{\omega_0^2 - \omega^2} E_0\right)^2$$

The flux of incident radiation is Flux $= cE_0^2$. So the cross section is

$$\frac{\text{Power}}{\text{Flux}} = \frac{e^2}{c^4}\left(\frac{e}{m}\frac{\omega^2}{\omega_0^2 - \omega^2}\right)^2 = \alpha^2 \left(\frac{\hbar}{mc}\right)^2 \left(\frac{\omega^2}{\omega_0^2 - \omega^2}\right)^2 = r_0^2 \left(\frac{\omega^2}{\omega_0^2 - \omega^2}\right)^2$$

(Note that the above may have missing factors of 2 and π. But it does otherwise have the same dependencies as the quantum mechanical version. Note that in the Thomson scattering limit, where $\omega \gg \omega_0$, the cross section is independent of wavelength.

Born approximation

Now we consider scattering of light with the formalism that we developed for determining scattering amplitudes from a fixed potential. The initial state has a single photon

$$|\, i\rangle = |\, i; k\lambda\rangle = a^\dagger_{k\lambda} |\, i; 0\rangle$$

and the final state

$$|\, f\rangle = |\, f; k'\lambda'\rangle = a^\dagger_{k'\lambda} |\, f; 0\rangle$$

The total energy $E = E_i + k = E_f + k'$. The collision rate is given by the golden rule

$$\frac{2\pi}{\hbar} |\, \langle f; k'\lambda' \mid T \mid i; k\lambda\rangle\, \rho_{k'\lambda'}$$

where

$$T = H_2 + H_1 \frac{1}{E - H_0 + i\epsilon} H_1$$

and $H_0 = H_\gamma + H_{matter}$ with no interaction.

$$H_2 = \frac{e^2}{2mc^2}|A|^2 \qquad H_1 = \frac{e}{m}\mathbf{p}\cdot\mathbf{A} - \mu\cdot\mathbf{B}$$

H_2 can create and annihilate (or the other way around). H_1 in second order can do the same. Just keeping the terms that can create k_2 and annihilate k_1 we get

$$\begin{aligned} |A|^2 &= \frac{e^2}{2mc^2}\sum_{k_1,k_2}\frac{1}{\sqrt{2Vk_1 2Vk_2}}\mathbf{e_{k_1}}\cdot\mathbf{e^*_{k_2}}e^{i(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{r}}(a_{k_1}a^\dagger_{k_2} + a^\dagger_{k_2}a_{k_1}) \\ &= \frac{1}{2V}\sum_k \frac{1}{k} + \frac{1}{V}\sum_{k_1 k_2}\frac{1}{\sqrt{k_1 k_2}}(\mathbf{e^*_{k_1}}\cdot\mathbf{e_{k_2}})e^{i(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{r}}a^\dagger_{k_2}a_{k_1} \end{aligned}$$