April 10, 2015
Lecture XXVIII

# Quantization of the E-M field

## 2.1 Lamb Shift revisited

We discussed the shift in the energy levels of bound states due to the vacuum fluctuations of the E-M fields. Our picture was that the fluctuating vacuum fields pushed the electron back and forth over a region of space defined by the strength of the fields and the electron mass. We estimate the uncertainty in the electron position due to the fields. The strength of the Coulomb potential that binds the electron to the proton depends of course on the position of the electron. In order to account for fluctuating position we average the potential over the position uncertainty. Far from the proton the difference of the potential averaged over this finite range of electron positions is zero. But at the origin of the potential, there is a finite contribution that will tend to reduce the effective coupling. At any rate we estimate the effect and found that because the vacuum fluctuations occur at all wavelengths, the position fluctuations diverge. We used a cutoff, excluding photons with energy greater than the electron rest energy.

Now we consider the effect in the language of transitions. Suppose that we have an electron in a bound state $A$. There is some amplitude for the electron to emit a photon with energy $\hbar ck$ and absorb it again. In the interim, energy is not necessarily conserved. There may be an intermediate bound state, but maybe not. The electron lives in a virtual state as does the photon. We write

$$\begin{aligned}
& \sum_k \sum_I \left\langle A \mid H_I^\dagger \mid I, \mathbf{k} \right\rangle \langle I, \mathbf{k} \mid H_I \mid A \rangle \\
= & \sum_k \sum_I \frac{e}{mc}\sqrt{\frac{\hbar c}{2Vk}} \langle A \mid a_k \epsilon^\lambda \cdot \mathbf{p} e^{i\mathbf{k}\cdot\mathbf{x}} \mid I, \mathbf{k} \rangle \frac{e}{mc}\sqrt{\frac{\hbar c}{2Vk}} \left\langle I, \mathbf{k} \mid a_k^\dagger \epsilon^\lambda \cdot \mathbf{p} e^{-i\mathbf{k}\cdot\mathbf{x}} \mid A \right\rangle \\
= & \sum_k \sum_I \left(\frac{e}{mc}\right)^2 \frac{\hbar c}{2Vk} \langle A \mid a_k \epsilon^\lambda \cdot \mathbf{p} e^{i\mathbf{k}\cdot\mathbf{x}} \mid I, \mathbf{k} \rangle \left\langle I, \mathbf{k} \mid a_k^\dagger \epsilon^\lambda \cdot \mathbf{p} e^{-i\mathbf{k}\cdot\mathbf{x}} \mid A \right\rangle
\end{aligned}$$

As we see in the above, all photon energies and polarizations are included. We are expecting that this interaction with the photons in the vacuum will result in a shift in the energy of the state $A$. The amplitudes $c_I$ and $c_A$ for the intermediate and initial state respectively are related according to

$$\begin{aligned}
i\hbar \dot{c}_I &= \sum_{photon} H_{IA} c_A e^{i(E_I - E_A)t/\hbar} \\
i\hbar \dot{c}_A &= \sum_{photon} \sum_I H_{AI} c_I e^{i(E_A - E_I)t/\hbar}
\end{aligned} \tag{2.1}$$

Here $H_{IA}$ corresponds to emission and $H_{AI}$ to absorption. These are the usual equations for the amplitudes to be in state $I$ where at $t = 0$, $c_A(0) = 1$ and $c_{I\neq A}(0) = 0$. We sum over all photon energies including those where $\hbar\omega \neq E_I - E_A$. Since we are looking for the shift in the energy of $E_A$ we try $c_A = e^{-i\Delta E_A t)/\hbar}$ so that

$$\psi_A(t) = \mid u_A \rangle e^{-i(E_A + \Delta E_A)t/\hbar}$$

The plan now is to substitute our guess for $c_A$ into the Equations 2.1, and solve for $\Delta E$. Since we are looking for an energy shift and not necessarily a transition rate, we will want eventually to integrate to $t \to \infty$.

$$\begin{aligned} c_I &= \frac{1}{i\hbar} \sum_{photons} H_{IA} \int_0^t e^{i(E_I - E_A - \Delta E_A + \hbar\omega)t'/\hbar} \\ c_I &= \sum_{photons} H_{IA} \frac{e^{i(E_I - E_A - \Delta E_A + \hbar\omega)t/\hbar} - 1}{(-E_I + E_A + \Delta E_A - \hbar\omega)} \end{aligned} \tag{2.2}$$

Next substitute Equation 2.2 and our guess for $c_A$ into the second of 2.1

$$\begin{aligned} \Delta E_A e^{-i(\Delta E_A t/\hbar)} &= \sum_{photon} \sum_I H_{AI} \sum_{photons} H_{IA} e^{-i\omega t} \frac{e^{i(-\Delta E_A + \hbar\omega)t/\hbar} - e^{i(E_A - E_I)t/\hbar}}{(-E_I + E_A + \Delta E_A - \hbar\omega)} \\ \Delta E_A &= \sum_{photon} \sum_I H_{AI} \sum_{photons} H_{IA} \frac{1 - e^{i(E_A - E_I + \Delta E_A - \hbar\omega)t/\hbar}}{(-E_I + E_A + \Delta E_A - \hbar\omega)} \end{aligned}$$

To lowest order in $\Delta E_A$ (drop the terms on the right).

$$\Delta E_A = \sum_{photon} \sum_I H_{AI} H_{IA} \frac{1 - e^{i(E_A - E_I - \hbar\omega)t/\hbar}}{(-E_I + E_A - \hbar\omega)} \tag{2.3}$$

We would like to evaluate that last equation for $\Delta E_A$ as $t \to \infty$ but it appears to oscillate. We note that

$$\begin{aligned} \lim_{t\to\infty} \frac{1 - e^{ixt}}{x} &= -\lim_{\epsilon\to 0} i \int_0^\infty e^{i(x+i\epsilon)t'} dt' \\ &= \lim_{\epsilon\to 0} \frac{1}{x + i\epsilon} \\ &= \lim_{\epsilon\to 0} \left[ \frac{x}{x^2+\epsilon^2} - \frac{i\epsilon}{x^2+\epsilon^2} \right] \\ &= \frac{1}{x} - i\pi\delta(x) \end{aligned}$$

Therefore, Equation 2.3 becomes

$$\Re\Delta E_A = \sum_{photon} \sum_I |H_{IA}|^2 \frac{1}{(-E_I + E_A - \hbar\omega)} \tag{2.4}$$

$$\mathrm{Im}\Delta E_A = -\pi \sum_{photon} \sum_I |H_{IA}|^2 \delta(E_A - E_I - \hbar\omega) \tag{2.5}$$

The real part of the energy shift is just that, a shift in the energy of the bound state $A$. It has contributions from all photon energies and intermediate states. The intermediate states are unrestricted. That is, $E_I > E_A$ is allowed. The electron is interacting with virtual photons in the vacuum. The imaginary part corresponds to the spontaneous decay rate from $I$ to $A$. The delta function enforces conservation of energy. Only photons with energy $E_I - E_A$ will contribute. Indeed there is an imaginary part only if there is a lower energy state available.

Therefore

$$-\frac{2}{\hbar}\mathrm{Im}[\Delta E_A] = \frac{1}{\tau_A} = \frac{\Gamma_A}{\hbar}$$

Then

$$\psi(t) = \mid u_A\rangle e^{-i(E_A + Re[\Delta E_A])t/\hbar} e^{-\Gamma_A t/2\hbar}$$

The probability of finding the state $A$

$$|\psi(t)|^2 \sim e^{-\Gamma t/\hbar}$$

Now consider the real part of the energy shift. We integrate over all of the photon phase space to account for the sum over photons.

$$\begin{aligned}
\Re\Delta E_A &= \sum_{photon}\sum_I |H_{IA}|^2 \frac{1}{(-E_I + E_A - \hbar\omega)} \\
&= \frac{c^2\hbar}{V}\left(\frac{e^2}{mc}\right)^2 \sum_I \int \frac{d^3kV}{(2\pi)^3}\frac{1}{2\omega}\sum_\lambda \frac{|(\mathbf{p}\cdot\epsilon^\lambda)_{IA}|^2}{E_A - E_I - \hbar\omega}
\end{aligned}$$

The sum over polarizations and the angular integration is the same as for our calculation of Rayleigh scattering, namely, $\int d\Omega \sum_\lambda |(\mathbf{p}\cdot\epsilon^\lambda)_{IA}|^2 = (8\pi/3)|(\mathbf{p})_{IA}|^2$. Then the real part

$$\Delta E_A = \frac{2}{3\pi}\left(\frac{e^2}{4\pi\hbar c}\right)^2 \frac{1}{(mc)^2}\sum_I \int \frac{E_\gamma |(\mathbf{p})_{IA}|^2 dE_\gamma}{E_A - E_I - \hbar\omega}$$

The integral diverges. If we choose a cutoff, $E_{max} = mc^2$, (certainly our non-relativstic wave functions) will not be valid for energies greater than the electron rest mass, then the energy shift is equivalent to our earlier calculation of the Lamb shift. This time we take it a step further and consider the energy shift of a free electron. After all, a free electron can similarly interact with virtual photons. The self-energy of the free electron manifests itself as a shift in the mass of the free electron. We are looking only for the change in the bound state energy due the real potential. So we may be double counting.

For the free electron, the difference is simply that there are no bound states. We integrate over all photon energies. Returning to equation 2.4 for the real part of the energy shift

$$\begin{aligned}
H_{IA} &= \left\langle \mathbf{p}', \mathbf{k} \mid \frac{e}{mc}\mathbf{p}\cdot A \mid \mathbf{p}\right\rangle \\
H_{IA} &= -\sqrt{\frac{\hbar c}{2kV}}\frac{e}{mc}\int \frac{e^{-i\mathbf{p}'\cdot\mathbf{x}/\hbar}}{\sqrt{V}} e^{-i\mathbf{k}\cdot\mathbf{x}}(\mathbf{p}\cdot\epsilon^\lambda)\frac{e^{i\mathbf{p}\cdot\mathbf{x}/\hbar}}{\sqrt{V}} d^3x
\end{aligned}$$

where the initial and final states of the electron are plane waves (momentum eigenstates) with $\mathbf{p}$ and $\mathbf{p}'$ respectively. The photon has energy $\hbar c\mathbf{k}$. Note that here we do not make the dipole approximation. By including the spatial dependence $e^{-i\mathbf{k}\cdot\mathbf{x}\cdot}$ in the integration $d^3x$ we end up with the momentum conserving delta function. For the energy shift of the bound state we did assume the dipole approximation. Are these results compatible? Then

$$H_{IA} = -\sqrt{\frac{\hbar}{2\omega}}\frac{e}{m}\frac{1}{V^{1/2}}\mathbf{p}\cdot\epsilon^\lambda \delta(\mathbf{p}' - \mathbf{p} + \hbar\mathbf{k}).$$

The denominator $E_A - E_I - \hbar\omega \rightarrow p^2/2m - (p - \hbar k)^2/2m - \hbar\omega \sim -\hbar\omega$. Then as before

$$\begin{aligned}
\Re\Delta E_A &= \sum_{photon}\sum_{I}|H_{IA}|^2\frac{1}{(-E_I + E_A - \hbar\omega)} \\
&= \frac{\hbar e^2}{2\omega V m^2}|\mathbf{p}\cdot\epsilon^\lambda|^2\int\frac{d^3kV}{(2\pi)^3\hbar\omega}\frac{1}{E_a - E_I - \hbar\omega}
\end{aligned}$$

We average $|\mathbf{p}\cdot\epsilon^\lambda|^2$ over initial state polarization and sum over final $\rightarrow \frac{8\pi}{3}\mathbf{p}^2$.

$$\begin{aligned}
\Delta E_A &= -\frac{\hbar}{2\omega}\frac{e^2}{m^2}\frac{\mathbf{p}^2}{V}\frac{V\int k^2 dk d\Omega}{(2\pi)^3\hbar\omega} \\
&= -\frac{8\pi}{3}\frac{\hbar}{2\omega}\frac{e^2}{m^2}\frac{\mathbf{p}^2}{V}\int\frac{d^3kV}{(2\pi)^3\hbar\omega} \\
&= -\frac{2}{3\pi}\frac{e^2}{4\pi\hbar c}\frac{1}{(mc)^2}\mathbf{p}^2\int\hbar d\omega \\
&= -\frac{2}{3\pi}\frac{e^2}{4\pi\hbar c}\frac{\mathbf{p}^2}{(mc)^2}\int dE_\gamma \\
&= C\mathbf{p}^2
\end{aligned}$$

Insofar as

$$\frac{1}{2m} = \frac{\partial E}{\partial\mathbf{p}^2}$$

we find that the observed mass

$$\frac{1}{2m_{obs}} = \frac{1}{2m_{bare}} + C \rightarrow m_{obs} = m_{bare}(1 - 2m_{bare}C)$$

assuming $C$ is small which it will be if we choose $E_{max} = mc^2$.

Our calculation of energy of state $A$ assumed the observed rather than the bare mass. We need to correct. Let's subtract the contribution for the free particle with momentum $A$. Note that $(\mathbf{p}^2)_{AA} = \sum_I|(\mathbf{p})_{IA}|^2$

$$\begin{aligned}
\Delta E_A^{observed} &= \frac{2}{3\pi}\left(\frac{e^2}{4\pi\hbar c}\right)\frac{1}{(mc)^2}\int\left(\sum_I\frac{E_\gamma|(\mathbf{p})_{IA}|^2}{E_A - E_I - \hbar\omega} + (\mathbf{p}^2)_{AA}\right)dE_\gamma \\
&= \frac{2}{3\pi}\left(\frac{e^2}{4\pi\hbar c}\right)\frac{1}{(mc)^2}\int\left(\sum_I\frac{|(\mathbf{p})_{IA}|^2(E_A - E_I)}{E_A - E_I - \hbar\omega}\right)dE_\gamma \\
&= \frac{2}{3\pi}\left(\frac{e^2}{4\pi\hbar c}\right)\frac{1}{(mc)^2}\left(\sum_I|(\mathbf{p})_{IA}|^2(E_I - E_A)\log\frac{E_\gamma^{max}}{E_A - E_I}\right)
\end{aligned}$$

The expression still diverges, but now logarithmically rather than linearly so much less sensitive to the cutoff. With some effort it is possible to show that

$$\sum_I|(\mathbf{p})_{IA}|^2(E_A - E_I) = -\frac{1}{2}\hbar\int|\psi_A|^2\nabla^2 V d^3x$$

[Let's try to do that. For the Hamiltonian $H_0 = \frac{\mathbf{p}^2}{2m} + V$,

$$\mathbf{p}H_0 - H_0\mathbf{p} = -i\hbar\nabla V$$

Next

$$\begin{aligned} \langle I \mid \mathbf{p}H_0 - H_0\mathbf{p} \mid A\rangle &= \langle I \mid -i\hbar\nabla V \mid A\rangle \\ \mathbf{p}_{IA}E_A - E_I\mathbf{p}_{IA} &= -i\hbar(\nabla V)_{IA} \end{aligned}$$

Multiply by $\mathbf{p}_{AI}$ and sum and note that the result must be real

$$\begin{aligned} \sum_I |\mathbf{p}_{IA}|^2(E_A - E_I) &= -i\hbar\sum \mathbf{p}_{AI}\cdot(\nabla V)_{IA} \\ &= i\hbar\sum(\nabla V)_{AI}\cdot\mathbf{p}_{IA} \\ &= -i\frac{\hbar}{2}\sum[\mathbf{p},(\nabla V)]_{AA} \\ &= -\frac{\hbar^2}{2}(\nabla^2 V)_{AA} \end{aligned}$$

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which for the hydrogen atom becomes $\nabla^2 V = e^2\delta(\mathbf{x})$, so

$$\sum_I |(\mathbf{p})_{IA}|^2(E_A - E_I) = -\frac{1}{2}e^2\hbar^2|\psi_A(0)|^2$$

The energy shift becomes

$$\begin{aligned} \Delta E_A^{observed} &= \frac{2}{3\pi}\left(\frac{e^2}{4\pi\hbar c}\right)\frac{1}{(mc)^2}\frac{1}{2}e^2\hbar^2|\psi_A(0)|^2\log\frac{E_\gamma^{max}}{\langle E_A - E_I\rangle_{ave}} \\ &= \frac{1}{3\pi}\alpha^2 4\pi\frac{\hbar^3c^3}{(mc^2)^2}|\psi_A(0)|^2\log\frac{E_\gamma^{\max}}{\langle E_A - E_I\rangle_{ave}} \\ &= \frac{1}{3\pi}\alpha^2 4\pi\frac{\hbar^3c^3}{(mc^2)^2}\frac{1}{\pi a_0^3}\log\frac{E_\gamma^{\max}}{\langle E_A - E_I\rangle_{ave}} \\ &= \frac{1}{3\pi}\alpha^2 4\pi\frac{\hbar^3c^3}{(mc^2)^2}\frac{(\alpha mc)^3}{\pi\hbar^3}\log\frac{E_\gamma^{\max}}{\langle E_A - E_I\rangle_{ave}} \\ &= \frac{4}{3\pi}\alpha^5 mc^2\log\frac{E_\gamma^{\max}}{\langle E_A - E_I\rangle_{ave}} \\ &= \frac{8}{3\pi}\alpha^3 E_{ryd}\log\frac{E_\gamma^{\max}}{\langle E_A - E_I\rangle_{ave}} \end{aligned}$$

If $E_\gamma^{max} = mc^2$ then

$$\Delta E_A^{observed} = 4.9\frac{8}{3\pi}\alpha^3 E_{ryd}$$