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Lecture III

1.2 Entangled States

If a composite system is in a pure state, its sub systems are in general in mixed states. Consider the two particle spin singlet state

$$| \alpha\rangle = \frac{1}{\sqrt{2}}(| +\rangle_1 | -\rangle_2 - | -\rangle_1 | +\rangle_2)$$

It is an entangled state. It cannot be written as a product of states of the two particles. The density matrix $\rho = | \alpha\rangle\langle\alpha |$. The expectation value of an observable that acts on only one particle ($\boldsymbol{\sigma_1}$ for example)

$$\langle\boldsymbol{\sigma_1}\rangle = \langle\alpha \mid \boldsymbol{\sigma_1} \mid \alpha\rangle = \frac{1}{2}\left[(\langle+ |_1\langle- |_2 - \langle- |_1\langle+ |_2)\boldsymbol{\sigma_1}(| +\rangle_1 | -\rangle_2 - | -\rangle_1 | +\rangle_2)\right]$$

$$= \frac{1}{2}\left[\langle+ |_1\boldsymbol{\sigma_1} | +\rangle_1 + \langle- |_1\boldsymbol{\sigma_1} | -\rangle_1\right]$$

which is the same as $\mathrm{Tr}\rho_1$ where

$$\rho_1 = \frac{1}{2}| +\rangle\langle- | + \frac{1}{2}| -\rangle\langle- |,$$

namely a completely mixed state. If our observables are in only a subsystem of the state, the state is mixed. Or put another way, if we only have access to one of the particles then there is no way to know whether or not it is in an entangled state. The measurements available to us will tell us nothing about the two particle state.

More generally, suppose $| \alpha_i\rangle$ and $| \beta_j\rangle$ are base states each for particle 1 and 2 and further that the base states are orthonormal. The simplest entangled state is a linear combination

$$c_1| \alpha_1\rangle| \beta_1\rangle + c_2| \alpha_2\rangle| \beta_2\rangle$$

$|c_1|^2 + |c_2|^2 = 1$. The density matrix for the pure state is

$$[c_1| \alpha_1\rangle| \beta_1\rangle + c_2| \alpha_2\rangle| \beta_2\rangle[\,[c_1\langle\alpha_1 |\langle\beta_1 | + c_2\langle\alpha_2 |\langle\beta_2 |]$$

The expectation value of an operator that acts only on particle 1, (perhaps measuring the spin of particle 1) is

$$\mathrm{Tr}(A\rho) = \sum_{ij}\langle\alpha_i |\langle\beta_j \mid [c_1| \alpha_1\rangle| \beta_1\rangle + c_2| \alpha_2\rangle| \beta_2\rangle[\,[c_1^*\langle\alpha_1 |\langle\beta_1 | + c_2^*\langle\alpha_2 |\langle\beta_2 |]\,A| \alpha_i\rangle| \beta_j\rangle$$

$$= \sum_{ij}\langle\alpha_i \mid [c_1| \alpha_1\rangle\delta_{j1} + c_2| \alpha_2\rangle\delta_{j2}[\,[c_1^*\langle\alpha_1 |\delta_{j1} + c_2^*\langle\alpha_2 |\delta_{j2}]\,A| \alpha_i\rangle$$

$$= \sum_{i}\langle\alpha_i \mid \left[|c_1|^2| \alpha_1\rangle\langle\alpha_1 | + |c_2|^2| \alpha_2\rangle\langle\alpha_2 |\right]A| \alpha_i\rangle$$

$$\rightarrow \rho_1 = |c_1|^2 |\ \alpha_1\rangle\langle\alpha_1\ | + |c_2|^2 |\ \alpha_2\rangle\langle\alpha_2\ |$$

It is clear that $\mathrm{Tr}\rho_1 = 1$. And furthermore

$$\rho_1^2 = |c_1|^4 |\ \alpha_1\rangle\langle\alpha_1\ | + |c_2|^4 |\ \alpha_2\rangle\langle\alpha_2\ |$$

And $\mathrm{Tr}(\rho_1^2) = |c_1|^4 + |c_2|^4 < 1$. Therefore the density matrix for particle 1 alone, that is assuming that when we measure anything about particle 1 alone we simply trace over the particle 2 states, corresponds to a mixed rather than a pure state. The entropy will depend on $|c_1|^2$ and $|c_2|^2$. If one or the other is zero than the initial state is pure but no longer entangled and the sub state is pure.

1.2.1 Hidden Variables

Now construct a theory with a hidden variable λ that determines the value of v_1 along $\hat{\mathbf{n}}$ to have value ± 1 that is given by $v_1(\lambda, \hat{\mathbf{n}}_1)$. The observed correlation will be

$$R(\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2) = \int v_1(\lambda, \hat{\mathbf{n}}_1) v_2(\lambda, \hat{\mathbf{n}}_2) w(\lambda) d\lambda$$

corresponding to

$$C(\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2) = \frac{1}{N} \sum_{v_1, v_2} N(v_1, v_2) v_1 v_2$$

where $w(\lambda)$ is the probability distribution of the λ. We want a local hidden variable theory. That means that $v_i(\lambda, \hat{\mathbf{n}}_i)$ depends only on $\hat{\mathbf{n}}_i$ where it is measured and not on the value of $\hat{\mathbf{n}}_2$. Also $w(\lambda)$ does not depend on the settings of the device that is doing the measuring. The values displayed by 1 cannot depend on the direction chosen for the measurement of 2. In the single state the correlations are perfect. Therefore

$$R(\hat{\mathbf{n}}, \hat{\mathbf{n}}) = -1$$

for all $\hat{\mathbf{n}}$. and

$$v_1(\lambda, \hat{\mathbf{n}}) = -v_2(\lambda, \hat{\mathbf{n}})$$

$$R(\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2) = -\int v_1(\lambda, \hat{\mathbf{n}}_1) v_1(\lambda, \hat{\mathbf{n}}_2) w(\lambda) d\lambda$$

Then

$$R(\hat{\mathbf{a}}, \hat{\mathbf{b}}) - R(\hat{\mathbf{a}}, \hat{\mathbf{c}}) = -\int \left[v_1(\hat{\mathbf{a}}, \lambda) v_1(\hat{\mathbf{b}}, \lambda) - v_1(\hat{\mathbf{a}}, \lambda) v_1(\hat{\mathbf{c}}, \lambda) \right] w(\lambda) d\lambda$$

Now $|v_1(\hat{\mathbf{b}})|^2 = 1$. Therefore

$$R(\hat{\mathbf{a}}, \hat{\mathbf{b}}) - R(\hat{\mathbf{a}}, \hat{\mathbf{c}}) = -\int \left[v_1(\hat{\mathbf{a}}) v_1(\hat{\mathbf{b}}) - v_1(\hat{\mathbf{b}}) v_1(\hat{\mathbf{b}}) v_1(\hat{\mathbf{a}}) v_1(\hat{\mathbf{c}}) \right] d\lambda$$

$$= -\int v_1(\hat{\mathbf{a}}) v_1(\hat{\mathbf{b}}) \left[1 - v_1(\hat{\mathbf{b}}) v_1(\hat{\mathbf{c}}) \right] d\lambda$$

The term in the square brackets is always greater than or equal to zero. Meanwhile $v_1(\hat{\mathbf{a}}) v_1(\hat{\mathbf{b}}) = \pm 1$. The maximum absolute value of the right had side is when that product is always the same, (either always +1 or always -1). Therefore the absolute value of the left hand side

$$|R(\hat{\mathbf{a}}, \hat{\mathbf{b}}) - R(\hat{\mathbf{a}}, \hat{\mathbf{c}})| \leq \int [1 - v_1(\hat{\mathbf{b}}) v_1(\hat{\mathbf{c}})] w(\lambda) d\lambda$$

$$|R(\hat{\mathbf{a}}, \hat{\mathbf{b}}) - R(\hat{\mathbf{a}}, \hat{\mathbf{c}})| \leq 1 + R(\hat{\mathbf{b}}, \hat{\mathbf{c}})$$

We computed earlier that $R(\hat{\mathbf{a}}, \hat{\mathbf{b}})$ for the spin singlet system, at least according to quantum mechanics is $Q(\hat{\mathbf{a}}, \hat{\mathbf{b}}) = -\cos\theta_{ab}$ If $\theta_{ab} = \theta_{bc} = \pi/4$ and $\theta_{ac} = \pi/2$ then the local hidden variable theory requires that

$$|-\cos(\pi/4) + \cos(\pi/2)| \leq 1 - \cos(\pi/4)$$

$$\frac{\sqrt{2}}{2} \leq 1 - \frac{\sqrt{2}}{2}$$

which ain't so. Evidently the observations consistent with quantum mechanics cannot be reproduced with a theory with a hidden variable.