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Lecture XXXI

Relativsitic Quantum Mechanics

3.1 Klein Gordon

Before we get to the Dirac equation, let's consider the most straightforward derivation of a relativistically invariant wave equation. The energy momentum relationship

$$E^2 = p^2 + m^2$$

becomes the Klein Gordon equation

$$(\frac{\partial^2}{\partial t^2} + \nabla^2)\psi = m^2\psi$$

on replacement $\mathbf{p} \rightarrow -i\hbar\nabla$ and $E \rightarrow i\hbar\frac{\partial}{\partial t}$. It is easy to write plane wave solutions, and indeed a free particle wave function had better be a solution to the KG equation. But perhaps there is more. After all, the solutions are scalar functions. Spin is not incorporated. Also, since it is second order in time, a unque solution requires specification of wave function and its first derivative at $t = 0$, whereas for the Schrodinger equation that is not required. There are negative energy solutions which are annoying. And consider the Maxwell equations. Electric and magnetic fields in empty space are solutions to a second order wave equation but E&B are totally independent. Turns out they are related by the first order Maxwell equations from which the wave equations are derived. Perhaps there is some equivalent first order relationship among the components of the particle spin that imply the Klein Gordon equation.

3.2 First order wave equation

We could just fish around with some guidance from dimensional analysis to come up with first order equations. Instead we will develop the Dirac equation as the Lorentz transformation of spinors. The str¡ategy comes from the notion that the existence of spin 1/2 was because it was possible to find a set of 2X2 matrices homomorphic to the rotation group in 3 real dimensions. Or altenratively, there exist a 2-dimensional representation of the generators of rotations as defined by certain commutation relations. We said that the generators had to behave according to

$$[J_i, J_j] = i\epsilon_{ijk}\hbar J_k$$

and sure enough the Pauli matrices had the appropriate commutation relation so we concluded that there is a 2-dimensional representation, and because the angular momentum algebra told us that eigenvalues of angular momentum eigenkets are integer or half integer, the 2-d representation corresponds to spin 1/2. Now we try to extend that strategy to Lorentz transformations. Rotations are a subgroup of the Lorentz group. First let's look again at rotations. Cast the usual 3-dimensional spatial vector into a 2X2 Hermitian matrix according to

$$X = \begin{pmatrix} -z & -x + iy \\ -x - iy & z \end{pmatrix} = -\mathbf{x} \cdot \sigma \tag{3.1}$$

The determinant of X is the rotation invariant length of the vector $-(z^2+x^z+y^2)$. If the matrix X is to represent the vector $\mathbf{x}$ then the transformation

$$X' = AXB$$

must preserve the determinant implying $\det AB = 1$. If Hermiticity is to be preserved then

$$X'^\dagger = X' = B^\dagger X A^\dagger \rightarrow A = B^\dagger$$

If the determinant of X is to remain invariant, then $\det X' = 1 \rightarrow \det AA^\dagger = 1 \rightarrow |\det A| = 1$. Also

$$\mathrm{Tr}X' = \mathrm{Tr}AXA^\dagger = \mathrm{Tr}XA^\dagger A \rightarrow$$

A unitary. We already know from our study of rotations that the most general 2X2 unitary matrix is

$$e^{i\hat{\mathbf{n}}\cdot\sigma\theta}$$

But if we had to construct it we could start with the most general 2X2 matrix expressed as $a+i\mathbf{b}\cdot\sigma$ where a and b are complex. If it is unitary, a and $\mathbf{b}$ are real. And if the absolute value of the determinant is 1, that leaves three free parameters, namely the three angles. And

$$A = \cos\frac{\theta}{2} + \sigma\cdot\hat{\mathbf{n}}\sin\frac{\theta}{2}$$

Next we define a 2 component vector spinor χ where under rotations $\chi' = A\chi$. Then the spinor formed as $X\chi$ transforms according to

$$X'\chi' = AX\chi = AXA^\dagger A\chi$$

We could also effect a spatial inversion. Then $X \rightarrow -X$. Note that $\det X = \det(-X)$ and that $(-X)$ transforms with the same A under rotations. We can not transform $\mathbf{x}\cdot\sigma \rightarrow -\mathbf{x}\cdot\sigma$ with A. A is the rotation operator and does not invert space. However, both vector and its inversion transform with the same A. Or the determinant is the same. Are all vectors with the same determinant connected by a rotation?

Now let's try to extend that strategy to transformations of 4-component vectors. The 2X2 representation of the 4-vector is

$$X_+ = \begin{pmatrix} t-z & -x+iy \\ -x-iy & t+z \end{pmatrix} = It - \mathbf{r}\cdot\sigma.$$

The invariant $Q = \det X$ as before. Again we search for a transformation that preserves the determinant. This time the trace is not an invariant. The $\det AA^\dagger = 1$, but A is not unitary. We now suppose that as before the product of spinor and X transforms according to

$$X'_+\chi'_+ \rightarrow A_+X_+\chi_+ = A_+X_+A^\dagger(A^\dagger)^{-1}\chi_+ \rightarrow \chi'_+ = (A^\dagger)^{-1}\chi$$

which is a bit awkward since $A^\dagger$ is in general not A^{-1}.

But now again consider the space inverted representation of the 4-vector.

$$X_+ = \begin{pmatrix} t-z & -x+iy \\ -x-iy & t+z \end{pmatrix} \rightarrow \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} = t + \mathbf{r}\cdot\sigma = X_-$$

The transformation for X_+ to X_- requires that

$$A(t - \mathbf{r}\cdot\sigma)A^\dagger = t + \mathbf{r}\cdot\sigma \to A\sigma A^\dagger = -\sigma$$

but there is no 2X2 matrix that along with its adjoint changes the sign of all three Pauli matrices. Just as for rotations we cannot effect space inversion by a proper Lorentz transformation. But unlike rotations, the Lorentz transformation for a vector and its inverted form are necessarily different. In order to include all possible 4-vectors with the same length, two 2X2 matrices are required, and therefore two sets of transformations and two independent spinors. Two 2X2 matrices X_+ and X_- are neccesary if we want to include reflections as Lorentz transformation of spinors. If a transformation of a spinor is to produce another spinor, then each needs to be represented as a combination of χ_+ and χ_-.

The second type of matrix X_- and spinor χ_- transform according to

$$A_- X_- A_-^\dagger = X'_-$$

and $\chi' = (A_-^\dagger)^{-1}$. The 4-component Dirac spinor is

$$\psi = \begin{pmatrix}\chi_+ \\ \chi_-\end{pmatrix}$$

χ_+ and χ_- are not mixed under proper Lorentz transformations (no reflection or time reversal. Note that time reversal has the same effect as reflection in that it cannot be accomplished by a proper Lorentz transformation.) Then

$$A = \begin{pmatrix}A_+ & 0 \\ 0 & A_-\end{pmatrix}.$$

Under reflection $\chi_+ \leftrightarrow \chi_-$. Then

$$\begin{pmatrix}\chi_- \\ \chi_+\end{pmatrix} = \begin{pmatrix}0 & I \\ I & 0\end{pmatrix}\begin{pmatrix}\chi_+ \\ \chi_-\end{pmatrix}$$

and

$$\psi \to \gamma_0\psi, \quad \gamma_0 = \begin{pmatrix}0 & I \\ I & 0\end{pmatrix}$$

Lorentz invariance including reflections requires a four component spinor. Irrespective of the dynamics. Just to include the possiblity that the reflected image of the state exists.

Let's construct A. We set a 2X2 matrix as $A_+ = a + i\mathbf{b}\cdot\sigma$ where $a, \mathbf{b}$ are complex. $\det AA^\dagger = 1 \to \det A = e^{i\phi} = \left|\begin{pmatrix}a + ib_z & ib_x + b_y \\ ib_x - b_y & a - ib_z\end{pmatrix}\right| = a^2 + b_z^2 + b_x^2 + b_y^2$ where ϕ is real. It turns out that ϕ is just an arbitrary phase since in the transformation $X' = AXA^\dagger$ the phase drops out. We can chose ϕ so that $a = \sqrt{1 - \mathbf{b}\cdot\mathbf{b}}$. Then $A = 1 + i\sin(\theta/2)\frac{\mathbf{b}}{|\mathbf{b}|}\cdot\sigma$ and we have two equations, one for the real part and the other for the imaginary part and that leaves 6 free parameters, namely the three angles and three velocities.

To construct A for a Lorentz transformation begin with boost along z. We will use rapidity as the boost parameters. The invariant $Q = (t - z)(t + z)$. If $(t - z) \to (t' - z') = (t - z)e^{-\xi}$ and $(t + z) \to (t' + z') = (t + z)e^{\xi}$ then $Q \to (t' - z')(t' + z') = (t - z)(t + z)$. We're not yet sure how ξ

is related to a velocity, but we do know that successive transformations first by ξ_1 and then ξ_2 are accomplished by $\xi = \xi_1 + \xi_2$. From the above we can write

$$\begin{aligned} t' - z' &= (t - z)e^{-}\xi \\ t' + z' &= (t + z)e^{+}\xi \\ t' &= t\cosh\xi + z\sinh\xi \\ z' &= t\cosh\xi + z\sinh\xi \end{aligned}$$

$$t' = t\sinh\xi + z\cosh\xi, \quad z' = t\sinh\xi + z\cosh\xi$$

with z, t in frame K and z', t' in frame K' that is moving with velocity $-v$ along z with respect to K. To connect with velocity note that if we are at rest in K and at $z = 0$ then

$$v = \Delta z'/\Delta t' = \tanh\xi.$$

Then

$$\frac{1}{\cosh^2\xi} = 1 - \tanh^2\xi = (1 - v^2) \rightarrow \cosh\xi = \frac{1}{\sqrt{1 - v^2}} = \gamma$$

and

$$\sinh\xi = \gamma v$$

So for a boost along z, $A = e^{-\xi\sigma_z/2}$. Then

$$X' = \begin{pmatrix} t' - z' & \\ 0 & t' + z' \end{pmatrix} = \begin{pmatrix} (t - z)e^{-\xi} & \\ 0 & (t + z)e^{\xi} \end{pmatrix} = e^{-\xi\sigma_z/2}(t - z\sigma_z)e^{-\xi\sigma_z/2}$$

Generalize to any direction and we have that

$$A_+ = e^{-\hat{\mathbf{n}}\cdot\sigma\xi/2}$$

To include rotations, $\hat{\mathbf{n}}$ is complex. What about Lorentz transformation of X_-. It had better turn out that

$$t' = t\cosh\xi + z\sinh\xi, z' = t\sinh\xi + z\sinh\xi$$

just like it does for X_+. Therefore, $A_- = e^{\hat{\mathbf{n}}\cdot\sigma}\xi$.

Finally

$$(A_+^\dagger)^{-1} = A^-$$

and we can write

$$\chi'_+ = A_-\chi_+, \quad \chi'_- = A_+\chi_-$$

3.2.1 Transformation to Dirac equation

We begin in the rest frame with a two-component spinor χ_0. We can effect the Lorentz transformation to a new frame using $A_\pm = e^{\mp\frac{1}{2}\xi\cdot\sigma}$. Then

$$\begin{aligned} \chi_+ &= e^{\frac{1}{2}\xi\hat{\mathbf{n}}\cdot\sigma}\chi_0 \\ \chi_- &= e^{-\frac{1}{2}\xi\hat{\mathbf{n}}\cdot\sigma}\chi_0 \\ \rightarrow & \\ \chi_- &= e^{-\xi\hat{\mathbf{n}}\cdot\sigma}\chi_+ = (\cosh\xi - \hat{\mathbf{n}}\cdot\sigma\sinh\xi)\chi_+ \\ \chi_+ &= e^{\xi\hat{\mathbf{n}}\cdot\sigma}\chi_- = (\cosh\xi + \hat{\mathbf{n}}\cdot\sigma\sinh\xi)\chi_- \end{aligned}$$

3.2.2 Lorentz group

The Lorentz transformation of a 4-vector

$$x_\mu \to \Lambda^\mu_\alpha x^\alpha$$

preserves the proper time

$$\tau^2 = g_{\mu\nu}x^\mu x^\nu = x_\mu x^\mu = \Lambda^\beta_\mu \Lambda^\mu_\alpha x_\beta x^\alpha$$

A boost along the x-axis is given by

$$\begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where the 4-vector is given by (t, x, y, z). An infinitesimal transformation in the x-direction is

$$\lambda^\mu_\alpha(\Delta\phi_x) = \delta^\mu_\alpha + \Delta\omega^\mu_\alpha$$

wnere

$$\Delta\omega^\mu_\alpha = \begin{pmatrix} 0 & -\Delta\phi_x & 0 & 0 \\ -\Delta\phi_x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since $ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dx_\mu dx^\mu$ is an invariant and

$$dx'_\mu dx^{\mu\prime} = \Lambda^\beta_\mu \Lambda^\mu_\alpha dx_\beta dx^\alpha$$

then

$$dx'_\mu dx^{\mu\prime} = (\delta^\beta_\mu + \Delta\omega^\beta_\mu)(\delta^\mu_\alpha + \Delta\omega^\mu_\alpha)dx_\beta dx^\alpha \tag{3.2}$$

and since $\Lambda^\beta_\mu \Lambda^\mu_\alpha = \delta^\beta_\alpha$ then from Equation 3.2 we get

$$\begin{aligned} (\Delta\omega^\beta_\alpha + \Delta\omega^\beta_\alpha)dx_\beta dx^\alpha &= 0 \\ \to (\Delta\omega_{\beta\alpha} + \Delta\omega_{\alpha\beta})dx^\beta dx^\alpha &= 0 \\ \to (\Delta\omega_{\beta\alpha} + \Delta\omega_{\alpha\beta}) &= 0 \end{aligned}$$

The contracted form of $\Delta\omega_{\alpha\beta}$ is antisummetric in its two indices. Then the infinitesimal boost along all three axes is

$$\Delta\omega_{\alpha\beta} = \begin{pmatrix} 0 & -\Delta\phi_x & -\Delta\phi_y & -\Delta\phi_z \\ \Delta\phi_x & 0 & 0 & 0 \\ \Delta\phi_y & 0 & 0 & 0 \\ \Delta\phi_z & 0 & 0 & 0 \end{pmatrix}$$

Adding infinitesimal rotations we have

$$\Delta\omega_{\alpha\beta} = \begin{pmatrix} 0 & -\Delta\phi_x & -\Delta\phi_y & -\Delta\phi_z \\ \Delta\phi_x & 0 & -\Delta\phi_z & \Delta\phi_y \\ \Delta\phi_y & \Delta\phi_z & 0 & -\Delta\phi_x \\ \Delta\phi_z & -\Delta\phi_y & \Delta\phi_x & 0 \end{pmatrix}$$

The generator for the boost in the x-direction is given by

$$K_x(\phi) = -i\frac{\partial \Lambda_x}{\partial \phi}|_{\phi=0} = -i\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and similarly

$$K_y(\phi) = -i\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_z(\phi) = -i\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

We can likewise extract the generators of rotations

$$J_x(\theta) = -i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad J_y(\theta) = -i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad J_z(\theta) = -i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Next evaluate the commutators and we find

$$\begin{aligned} [K_x, K_y] &= -iJ_z \\ [J_x, K_y] &= iK_z \\ [J_x, J_y] &= iJ_z \end{aligned}$$

The generators of rotations (angular momentum operators) do not generate boosts. They transform among themselves. Rotations are a subgroup of the Lorentz group. The generators of boosts mix with rotations. The product of two boosts corresponds to a rotation. The boosts do not form a subroup. But we can create a pair of subgroups by defining

$$M_n^{\pm} = \frac{1}{2}(J_n \pm iK_n)$$

Then

$$\begin{aligned} [M_i^+, M_j^+] &= i\epsilon_{ijk}M_k^+ \\ [M_i^-, M_j^-] &= i\epsilon_{ijk}M_k^- \\ [M_i^+, M_j^-] &= 0 \end{aligned}$$

Each of the M^+ and M^- forms a subgroup. The groups are distinct, corresponding to the two distinct types of spinors, namely χ_+ and χ_-. The transformation in the 2D complex vector space is determined by exponentiating the generator. The generator of the boost has opposite sign for $M^{\pm}$, so the transformation for the spinor become

$$e^{\pm\frac{1}{2}\xi\hat{\mathbf{n}}\cdot\sigma}$$