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Lecture XXXII

Relativsitic Quantum Mechanics

3.0.1 Free particle Dirac equation

Now we know how a spinor transforms. Suppose we begin with a spin 1/2 particle at rest in frame K, represented as ζ. Then it frame K', which is boosted along z with respect to K, it would have momentum $p_z = v\gamma m$ and look like

$$\chi'_+ = A_-\zeta = e^{\sigma_z\xi/2}\zeta$$

Alternatively we might represent the state with momentum p_z by

$$\chi'_- = A_+\zeta = e^{-\sigma_z\xi/2}\zeta$$

In the rest frame the vector and the parity reversed vector are identical. χ_+ and χ_- are evidently related. One is the mirror image of the other. A Lorentz transformation brings one into the other and

$$\begin{aligned}
e^{-\sigma_z\xi}\chi_+ &= \chi_- \\
(\cosh\xi - \sigma_z \sinh\xi)\chi_+ &= \chi_- \\
m(\cosh\xi - \sigma_z \sinh\xi)\chi_+ &= m\chi_- \\
mc^2(\gamma - \sigma_z\gamma(v/c))\chi_+ &= mc^2\chi_- \\
(E - c\sigma_z p_z)\chi_+ &= mc^2\chi_- = (E - c\sigma_z p_z)\chi_+
\end{aligned}$$

and

$$(E + \sigma_z p_z)\chi_- = m\chi_+$$

And since there is nothing special about z

$$\begin{aligned}
(E - c\sigma\cdot\mathbf{p})\chi_+(\mathbf{p}) &= mc^2\chi_-(\mathbf{p}) \\
(E + c\sigma\cdot\mathbf{p})\chi_-(\mathbf{p}) &= mc^2\chi_+(\mathbf{p})
\end{aligned}$$

These are the Dirac equations for a free particle with momentum $\mathbf{p}$. For a massless particle they reduce to the Weyl equations

$$\begin{aligned}
(E - c\sigma\cdot\mathbf{p})\chi_+(\mathbf{p}) &= 0 \\
\rightarrow (1 - \sigma\cdot\hat{\mathbf{p}})\chi_+(\mathbf{p}) &= 0 \\
(E + c\sigma\cdot\mathbf{p})\chi_-(\mathbf{p}) &= 0 \\
\rightarrow (1 + \sigma\cdot\hat{\mathbf{p}})\chi_-(\mathbf{p}) &= 0
\end{aligned}$$

We see that $\chi_\pm$ are eigenstates of helicity. The Weyl equations are only coupled by mass, not by electromagnetic interactions. In the ulra-relativistic regime, the Dirac equations reduce to the two uncoupled Weyl equations that conserve helicity.

To get to the Dirac equations in coordinate space

$$\mathbf{p} \to -i\hbar\nabla, \quad E \to i\hbar\frac{\partial}{\partial t}$$

Then the Dirac equations are

$$\begin{aligned}(\hbar\frac{\partial}{\partial t} + c\hbar\sigma\cdot\nabla)\phi_+(\mathbf{r},t) &= -imc^2\phi_-(\mathbf{r},t)\\ (\hbar\frac{\partial}{\partial t} - c\hbar\sigma\cdot\nabla)\phi_-(\mathbf{r},t) &= -imc^2\phi_+(\mathbf{r},t)\end{aligned}$$

where

$$\phi_\pm(\mathbf{r},t) = \int d^3p e^{-iEt} e^{i\mathbf{p}\cdot\mathbf{r})}\chi_\pm(\mathbf{p})$$

Both $\phi_\pm$ satisfy the Klein Gordon equation on multiplying by $(\frac{\partial}{\partial t} \mp c\sigma\cdot\nabla)$

3.1 4-component spinor

In the interest of compactness we define a 4-component spinor

$$\psi = \begin{pmatrix}\phi_{+1}\\ \phi_{+2}\\ \phi_{-1}\\ \phi_{-2}\end{pmatrix} = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}$$

and

$$\Sigma_i = \begin{pmatrix}\sigma_i & 0\\ 0 & -\sigma_i\end{pmatrix}$$

Then we can write

$$\left(\hbar\frac{\partial}{\partial t} + c\hbar\Sigma^i\frac{\partial}{\partial x^i}\psi + imc^2\gamma_0\right)\psi = 0$$

Also define

$$\gamma^i = \begin{pmatrix}0 & -\sigma_i\\ \sigma_i & 0\end{pmatrix} = \gamma_0\Sigma_i = -\gamma_i$$

Then the differential operator

$$\gamma^0\frac{1}{c}\frac{\partial}{\partial t} + \gamma^i\frac{\partial}{\partial x^i} = \gamma^0\partial_0 + \gamma\cdot\nabla = \gamma^\mu\partial_\mu$$

where

$$\frac{\partial}{\partial x^\mu} = \partial_\mu = (\frac{1}{c}\partial_t, \nabla), \qquad \partial x_\mu = \partial^\mu = (\frac{1}{c}\partial_t, -\nabla)$$

Then the Dirac equation is

$$(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0$$

The four 4X4 Dirac matrices transform like a four vector in Minkowski space just as the Pauli matrices transform like a 3 vector in Euclidean space.

3.2 Electromagnetic Interacton

We introduce the electromangetic interaction just as for nonrelativistic quantum mechanics.

$$\mathbf{p} \rightarrow \mathbf{p} - \frac{e}{c}\mathbf{A}, \quad E \rightarrow E - eV$$

or in four vector notation

$$p^\mu \rightarrow p^\mu - \frac{e}{c}A^\mu$$

where $A^\mu = (V, \mathbf{A})$. $(A_\mu = V, -\mathbf{A})$. Then $i\hbar\partial_\mu \rightarrow i\hbar\partial_\mu - \frac{e}{c}A_\mu$ and the Dirac equation with electromagnetic fields becomes

$$(i\hbar\gamma^\mu(\frac{\partial}{\partial x_\mu} + i\frac{e}{\hbar c}A_\mu) - mc)\psi = 0$$

The same follows from our insistence on local phase invariance

$$\psi \rightarrow \psi e^{-i\frac{e}{\hbar c}\lambda(x)}$$

Since

$$\psi(\frac{\partial}{\partial x_\mu} + i\frac{e}{\hbar c}A_\mu)e^{-i\frac{e}{\hbar c}\lambda(x)}\psi = e^{-i\frac{e}{\hbar c}\lambda(x)}(i\frac{\partial}{\partial x_\mu} - \frac{e}{\hbar c}(A_\mu + \frac{\partial}{\partial x_\mu}\lambda)\psi$$

the Dirac equation is invariant as long as we make the gauge transformation

$$A_\mu \rightarrow A_\mu - \frac{\partial}{\partial x_\mu}\lambda$$

We see that in the ultra-relativistic regime, where $m \rightarrow 0$, that for solutions to the Dirac equation in the presence of an electromagnetic field helicity is an invariant. This is most readily evident by writing the momentum space version of the Dirac equation that we started with

$$\begin{aligned}
(E - c\sigma \cdot \mathbf{p})\chi_+(\mathbf{p}) &= mc^2\chi_-(\mathbf{p}) \\
\rightarrow (E - eV - c\sigma \cdot (\mathbf{p} - \frac{e}{c}\mathbf{A}))\chi_+(\mathbf{p}) &= mc^2\chi_-(\mathbf{p}) \\
(E + c\sigma \cdot \mathbf{p})\chi_-(\mathbf{p}) &= mc^2\chi_+(\mathbf{p}) \\
\rightarrow (E - eV + c\sigma \cdot (\mathbf{p} - \frac{e}{c}\mathbf{A}))\chi_-(\mathbf{p}) &= mc^2\chi_+(\mathbf{p})
\end{aligned}$$

Clearly in the limit of zero mass, the left and right handed spinors are decoupled. Suppose there was a Lorentz invariant scalar field?

3.3 Non-relativistic limit

In the rest frame of the particle there is no difference between the spinors χ_+ and χ_-. In the rest frame there is no handedness, suggesting the linear combinations

$$\Psi = \frac{1}{\sqrt{2}}(\phi_+ + \phi_-) \quad \Phi = \frac{1}{\sqrt{2}}(\phi_+ - \phi_-)$$

Write the momenum space version of the Dirac equatons included electromagnetic interactions

$$\begin{aligned}(P^0 - \mathbf{P}\cdot\sigma)\phi_+ &= mc\phi_- \\ (P^0 + \mathbf{P}\cdot\sigma)\phi_- &= mc\phi_+\end{aligned}$$

where $P^0 = \frac{E-eV}{c}$, $\mathbf{P} = \mathbf{p} - \frac{e}{c}\mathbf{A}$. Taking the sum and difference of the two equations we find

$$P^0\Psi - \mathbf{P}\cdot\sigma\Phi = mc\Psi \tag{3.1}$$

$$P^0\Phi - \mathbf{P}\cdot\sigma\Psi = -mc\Phi \tag{3.2}$$

In the low energy limit $E - eV = mc^2 + E_{nr} - eV$, where E_{nr} is the non relativistic energy. Then Equation 3.2 becomes

$$c\mathbf{P}\cdot\sigma\Psi = (E_{nr} - eV + 2mc^2)\Phi$$

$$\rightarrow \frac{1}{(E_{nr} - eV + 2mc^2)} c\mathbf{P}\cdot\sigma\Psi = \Phi \tag{3.3}$$

and Equation 3.1

$$c\mathbf{P}\cdot\sigma\Phi = (E_{nr} - eV)\Psi$$

Substitution of Equation 3.3 into 3.1 gives

$$c(\mathbf{P}\cdot\sigma)\frac{1}{(E_{nr} - eV + 2mc^2)} c\mathbf{P}\cdot\sigma\Psi = (E_{nr} - eV)\Psi$$

$$\frac{c\mathbf{P}\cdot\sigma}{(2mc^2)}(1 + \frac{eV - E_{nr}}{2mc^2})c\mathbf{P}\cdot\sigma\Psi = (E_{nr} - eV)\Psi \tag{3.4}$$

To lowest order in v/c

$$(\frac{(\mathbf{P}\cdot\sigma)^2}{(2m)} + eV)\Psi = E_{nr}\Psi \tag{3.5}$$

Then

$$\begin{aligned}(\mathbf{P}\cdot\sigma)^2 &= \sum_{i,j} P_i\sigma_i P_j\sigma_j = P^2 + i\epsilon_{ijk}[P_i, P_j]\sigma_k \\ &= P^2 + i\sigma\cdot(\mathbf{P}\times\mathbf{P}) \\ &= P^2 + i\sigma\cdot(\mathbf{p} - \frac{e}{c}\mathbf{A})\times(\mathbf{p} - \frac{e}{c}\mathbf{A}) \\ &= P^2 - i\frac{e}{c}\sigma\cdot(\mathbf{p}\times\mathbf{A} + \mathbf{A}\times\mathbf{p}) \\ &= P^2 - i\frac{e}{c}\sigma\cdot(-i\hbar\nabla\times\mathbf{A} - \mathbf{A}\times\mathbf{p} + \mathbf{A}\times\mathbf{p}) \\ &= P^2 - i\frac{e}{c}\sigma\cdot(-i\hbar\mathbf{B})\end{aligned}$$

Finally 3.5 becomes

$$\begin{aligned}
\left(\frac{p^2 - i\frac{e}{c}\sigma\cdot(-i\hbar\mathbf{B})}{2m} + eV\right)\Psi &= i\hbar\frac{\partial}{\partial t}\Psi \\
\left(\frac{p^2}{2m} - \frac{e\hbar}{2mc}\sigma\cdot\mathbf{B} + eV\right)\Psi &= i\hbar\frac{\partial}{\partial t}\Psi
\end{aligned}$$

the Schrodinger equation for a charged particle with magnetic moment $\mu = \frac{e\hbar}{2mc}\sigma$.