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Lecture IV

1.1.1 Bell Inequality - Spin correlation

Consider the final spin singlet state of the decay $\eta^0 \rightarrow \mu^+\mu^-$

We suppose that the η^0 decays and the muon and μ^+ travel in opposite directions, with equal but opposite momenta, and equal but opposite spin. Now we can measure the component of the muon and/or μ^+ spin along any axis we like. If we measure the muon spin to be $+\frac{1}{2}$ along the $\hat{a}$ direction then we are guaranteed to measure $-\frac{1}{2}$ along the $\hat{a}$ direction for μ^+s.

The muon and μ^+, which we will label particles 1 and 2 respectively are in a spin singlet state

$$| \psi\rangle = \frac{1}{\sqrt{2}}\left(| +\frac{1}{2}\rangle_1 | -\frac{1}{2}\rangle_2 - | -\frac{1}{2}\rangle_1 | +\frac{1}{2}\rangle_2 \right)$$

And this representation is totally independent of your choice for the direction of the z-axis. A measurement of the spin of either particle along any axis will yield $\pm\frac{1}{2}$ with equal likelihood. There is no preferred direction in space because the two particle state has zero total angular momentum. On the other hand, once we have measured the μ^- spin, the result of a measurement of the μ^+ spin along the same axis is known with certainty before the measurement is made. The probability of a particular outcome along any other axis can be calculated. The two particle wave function collapses with the measurement of the μ^-. Now we know precisely the spin of the μ^+, (at least along the axis that we measured the spin of the μ^-). The collapse of the wave function propagates instantaneously from the point in space where we measure the muon spin to the location of the μ^+. You might be going about your business measuring the spin of the μ^+ along the $\hat{b}$ direction. The average of product of your measurements will be zero, $+\frac{1}{2}$ and $-\frac{1}{2}$ with equal likelihood. Then someone else decides to measure the spin of the μ^- along the $\hat{b}$ direction, and having done so can now predict with absolute certainty the result of each of your μ^+ measurements (along that same direction). Spooky action at a distance.

A possible way out of this action at a distance, or instantaneous propagation of the collapsing wave function dilemma is to suppose that the outcome of the measurements of the spin along any particular direction was determined at the moment of η^0 decay. That is at the birth of the muon and μ^+. We are unable to predict the outcome of the measurement, but that may simply reflect our ignorance, and the inadequacy of our theory. Who says that the information is not all along available to the muon. In this scenario, there is some hidden variable, something that describes the μ^- and μ^+ above and beyond the quantum mechanical wave function, that carries the information as to the outcome of any spin measurement.

1.1.2 Realist vs Orthodox view

The realist's objection is the notion of the instantaneous collapse of the wave function everywhere in space suggests an influence propagating through space faster than light. The realist does not necessarily reject the predictions of quantum mechanics, but rejects the notion that it is the measurement itself that determines the state of the system. If the outcome of the measurement of the spin of the muon (μ^+) were determined in advance of the measurement then there is no problem. Then the correlation does not depend on the measurement itself. It is really only our ignorance and the incompleteness of our theory that makes it look as though there is action at a distance

Until Bell came along in 1964, it did not matter. How could one possibly tell whether the μ^- and μ^+ spins were determined at birth or by the act of measurement? There was always perfect correlation. Spin $+\frac{1}{2}$ measured for the μ^- along a particular direction always corresponded to spin $-\frac{1}{2}$ along that same direction for the μ^+.

Then Bell showed that the realist and orthodox theories predicted a measurable difference.

1.2 Three measurement test

Let's imagine an experiment in which we can measure the spins of the μ^- and μ^+s along three possible directions, $\hat{a}$ is in the +z direction, $\hat{b}$ and $\hat{c}$ are at $\pm 120°$ respectively. An event is the decay of the η^0 into μ^- and μ^+. For each event we measure the spin of the μ^- along one of the 3 directions chosen at random and the spin of the μ^- along one of the 3 directions also chosen at random. There are a total of nine measurements, $(\hat{\mathbf{a}}^+, \hat{\mathbf{a}}^-), (\hat{\mathbf{a}}^+, \hat{\mathbf{b}}^-), (\hat{\mathbf{a}}^+, \hat{\mathbf{c}}^-), (\hat{\mathbf{b}}^+, \hat{\mathbf{a}}^-), (\hat{\mathbf{b}}^+, \hat{\mathbf{b}}^-), (\hat{\mathbf{b}}^+, \hat{\mathbf{c}}^-), (\hat{\mathbf{c}}^+, \hat{\mathbf{a}}^-), (\hat{\mathbf{c}}^+, \hat{\mathbf{b}}^-), (\hat{\mathbf{c}}^+, \hat{\mathbf{c}}^-)$. Furthermore, the measurement directions are not chosen until after the η^0 has decayed. There is no way that the μ^- and μ^+ can know at birth in what direction their spins will eventually be measured. Then we can determine the average of the product of the measured spins, namely their correlation. That will be

$$P(\hat{a},\hat{a}) + P(\hat{a},\hat{b}) + P(\hat{a},\hat{c}) + P(\hat{b},\hat{a}) + P(\hat{b},\hat{b}) + P(\hat{b},\hat{c}) + P(\hat{c},\hat{a}) + P(\hat{c},\hat{b}) + P(\hat{c},\hat{c})$$

where $P(\hat{\mathbf{a}}, \hat{\mathbf{b}})$ is the production of the expectation values of the spins measured along $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ Quantum mechanics predicts that $P(\hat{\mathbf{a}}, \hat{\mathbf{b}}) = -\hat{\mathbf{a}} \cdot \hat{\mathbf{b}}$. The average of all 9 measurements will be

$$\begin{aligned}
\langle P \rangle &= \left(-1 - \cos\frac{2\pi}{3} - \cos\frac{-2\pi}{3} - \cos\frac{2\pi}{3} - 1 - \cos\frac{-2\pi}{3} - \cos\frac{2\pi}{3} - \cos\frac{-2\pi}{3} - 1\right) \\
&= \left(-3 - 6\cos\frac{2\pi}{3}\right) \\
&= \left(-3 + 6(\frac{1}{2})\right) \\
&= 0
\end{aligned}$$

That is, the expectation value of the product of the spins averaged over the nine measurements is zero

Next we figure out what happens if the outcome of the measurements is predetermined, supposing that when each muon is born, it is labeled with an outcome for all possible measurements. There are 8 possible sets of labels for μ^-, namely

$$(+,+,+),(+,+,-),(+,-,+),(+,-,-),(-,+,+),(-,+,-),(-,-,+),(-,-,-)$$

where $(+,+,+)$ means that measurements of the μ^- spin along $\hat{a}, \hat{b}$, or $\hat{c}$ will all yield +1. $(+,+,-)$ means that measurements along $\hat{a}$ and $\hat{b}$ will be +1 and $\hat{c}$ will be −1 and so on. Of course if the μ^+ assignments are $(+,+,+)$ then the μ^+ assignments must be $(-,-,-)$ so that if the spins of both particles are measured along the same direction, the measured values will be equal and opposite.

Now we have no idea how many of each of the eight will be created. That depends on the theory. But we can nevertheless say something about the outcome of the measurements. For each type of muon (1-8) there are 9 distinct measurements. The 9 measurements for types 1 and 2 are listed

Table 1.1: Measurements and outcomes for muon types 1 and 2

Configuration	Muon	μ^+	$\sigma_- \cdot \sigma_+$	$\langle \sigma_- \cdot \sigma_+ \rangle$
1	$\hat{a}+$	$\hat{a}-$	-1	-1
1	$\hat{a}+$	$\hat{b}-$	-1	
1	$\hat{a}+$	$\hat{c}-$	-1	
1	$\hat{b}+$	$\hat{a}-$	-1	
1	$\hat{b}+$	$\hat{b}-$	-1	
1	$\hat{b}+$	$\hat{c}-$	-1	
1	$\hat{c}+$	$\hat{a}-$	-1	
1	$\hat{c}+$	$\hat{b}-$	-1	
1	$\hat{c}+$	$\hat{c}-$	-1	
2	$\hat{a}+$	$\hat{a}-$	-1	$-\frac{1}{9}$
2	$\hat{a}+$	$\hat{b}-$	-1	
2	$\hat{a}+$	$\hat{c}+$	$+1$	
2	$\hat{b}+$	$\hat{a}-$	-1	
2	$\hat{b}+$	$\hat{b}-$	-1	
2	$\hat{b}+$	$\hat{c}+$	$+1$	
2	$\hat{c}-$	$\hat{a}-$	$+1$	
2	$\hat{c}-$	$\hat{b}-$	$+1$	
2	$\hat{c}-$	$\hat{c}+$	-1	
...	...	...	...	

in Table 1. For configurations 1 and 8 the average of the product of spins is -1. For all of the rest it is $-\frac{1}{9}$. We know that the muon and μ^+ will be identified with some combination of the 8 configurations listed. There are no other possibilities. We don't know how much of each. But no matter what the weighting, it must be that $-1 \le \langle \sigma_- \cdot \sigma_+ \rangle \le -\frac{1}{9}$. Meanwhile, we already figured out that according to quantum mechanics, the average is zero.

So we do the measurement and find that the results are consistent with quantum mechanics and inconsistent with a theory in which the outcome of the measurements is determined before the measurement is made.

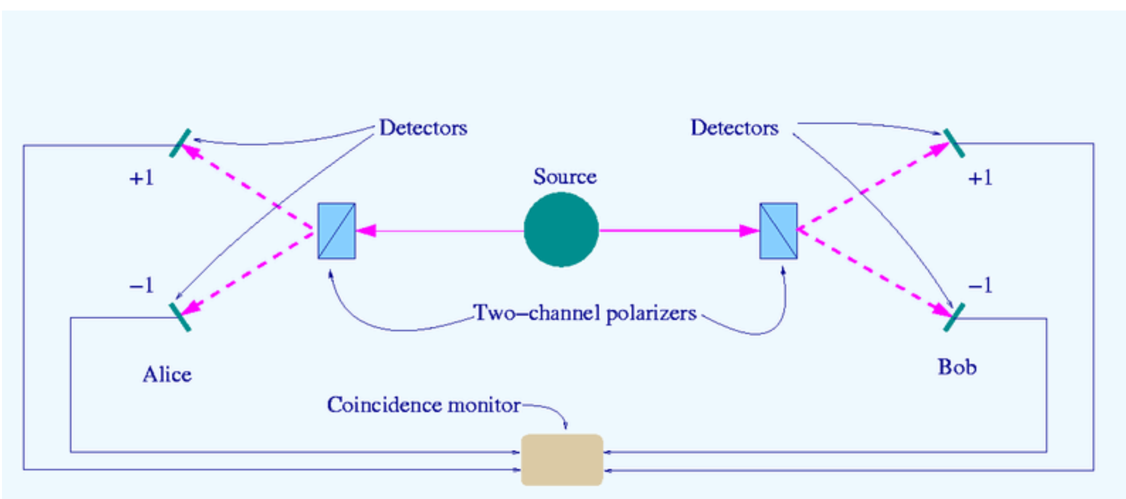
1.2.1 Instantaneous collapse of the wave function

Does the measurement of the μ^- influence the outcome of the μ^+ measurement? Evidently. Does the measurement of the μ^- cause a particular outcome for the μ^+ measurement? The muon measurer has no control over the outcome of the muon measurement. He cannot make a given muon spin up or down.

The experimenter at the μ^- detector might decide not to make a measurement at all. But the μ^+ measurer will never know. Whether or not the muon is measured, the μ^+ data is random. The separate lists of measurements considered separately are random, independently. Only on comparison is there a correlation. The act of measuring μ^- does not make the μ^+ measurements less random.

In another Lorentz frame, the μ^+ measurement occurs first and she is no more causing the μ^- to have a particular spin than the other way around.

1.2.2 Test of Bell's Theorem with polarized photons



The source S produces pairs of photons sent in opposite directions. Each photon encounters a two-channel polarizer whose orientation (a or b) can be set uby the experimenter. Emerging signals from each channel are detected and coincidences of 4 types (++,–,+-,-+) counted.

1.2.3 Clauser Horne Inequality

The Bell inequality is difficult to test experimentally. The derivation assumed perfect anticorrelation,namely $C(\hat{\mathbf{n}}, \hat{\mathbf{n}}) = -1$. Clauser and Horne derived an alternative inequality that can be understood in terms of a counting experiment.

Suppose that some observable of the two particles registers as a count in a detector. In the case of the muons we might arrange our Stern Gerlach magnet that is oriented at some angle, so that it kicks spin $+1$ into the detector. If the composite state consists of two photons, the detector registers a hit if the polarization is along some direction. The inequality will be determined by counting. There will be a total of N events, with $N_1(\hat{\mathbf{a}})$ counts in detector 1 when it is set to select $\hat{\mathbf{a}}$ and $N_2(\hat{\mathbf{b}})$ counts in detector 2 when it is set to select $\hat{\mathbf{b}}$. The number of coincidences of the two detectors with settings $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ respectively is $N_{12}(\hat{\mathbf{a}}, \hat{\mathbf{b}})$. The probabilities are

$$p_1(\hat{\mathbf{a}}) = N_1(\hat{\mathbf{a}})/N \quad p_2(\hat{\mathbf{b}}) = N_2(\hat{\mathbf{b}})/N \quad p_{12}(\hat{\mathbf{a}}, \hat{\mathbf{b}})) = N_{12}(\hat{\mathbf{a}}, \hat{\mathbf{b}})/N$$

In this formulation, the hidden variable will determine the probability that $p_1(\hat{\mathbf{a}})$ will have a certain value. Remember that in the Bell formulation, the hidden variable determined absolutely the value of the spin for a particular measurement. Then

$$p_1(\hat{\mathbf{a}}) = \int (p_a(\hat{1}, \lambda) w(\lambda) d\lambda$$

and

$$p_{12}(\hat{\mathbf{a}}, \hat{\mathbf{b}}, \lambda) = p_1(\hat{\mathbf{a}}, \lambda) p_2(\hat{\mathbf{b}}, \lambda)$$

and

$$p_{12}(\hat{\mathbf{a}}, \hat{\mathbf{b}}) = \int p_1(\hat{\mathbf{a}}, \lambda) p_2(\hat{\mathbf{b}}, \lambda) w(\lambda) d\lambda$$

It can be shown that for any four real numbers x, x', y, y' in the range $0 \leq r \leq 1$ that

$$xy - xy' + x'y + x'y' \leq x' + y$$

If we identify the probabilities (all in the range between 0 and 1) as

$$x = p_1(\hat{\mathbf{a}}, \lambda), \quad y = p_2(\hat{\mathbf{b}}, \lambda), \quad x' = p_1(\hat{\mathbf{a}}', \lambda), \quad y' = p_2(\hat{\mathbf{b}}', \lambda)$$

and subsitute into our inequality we get

$$p_1(\hat{\mathbf{a}}, \lambda) p_2(\hat{\mathbf{b}}, \lambda) - p_1(\hat{\mathbf{a}}, \lambda) p_2(\hat{\mathbf{b}}', \lambda) + p_1(\hat{\mathbf{a}}', \lambda) p_2(\hat{\mathbf{b}}, \lambda) + p_1(\hat{\mathbf{a}}', \lambda) p_2(\hat{\mathbf{b}}', \lambda) \leq p_1(\hat{\mathbf{a}}', \lambda) + p_2(\hat{\mathbf{b}}, \lambda)$$

Next multiply by $w(\lambda)$ and integrate over all λ to get

$$p_{12}(\hat{\mathbf{a}}, \hat{\mathbf{b}}) - p_{12}(\hat{\mathbf{a}}, \hat{\mathbf{b}}') + p_{12}(\hat{\mathbf{a}}', \hat{\mathbf{b}}) + p_{12}(\hat{\mathbf{a}}', \hat{\mathbf{b}}') \leq p_1(\hat{\mathbf{a}}') + p_2(\hat{\mathbf{b}})$$

This is the Clauser Horne Inequality. Note that it does not depend on any specific correlations of spin states or perfect anti-correlation.

1.2.4 Experimental Test with 2 photons

An atomic s-state with total angular momentum zero and even parity decays in two steps. Photon γ_1 is emitted in the E1 transition from the S-state to a P-state with $m = \pm 1, 0$. Photon γ_2 is emitted in the second E1 transition to the ground state. The final state of the atom also has zero angular momentum and even parity. Therefore the photons which are emitted back to back have the same helicity, so that their total angular momentum is zero. The two photons have differnt energies, $\omega - 1$ and ω_2. The helicity of each of the photons is determined by the intermediate state. If the intermediate state is $m = +1$ then the helicity of both photons is odd and if $m = -1$ then the helicities are even. The energy of the intermediate state is degenerate. There is no magnetic field that might split the energies of the $m = \pm 1, 0$ levels. The final pure photon state is therefore the linear combination

$$| \alpha\rangle = \frac{1}{\sqrt{2}} (| +1\rangle| +1\rangle + | -1\rangle| -1\rangle)$$

where the states $| \pm 1\rangle$ are the right and left handed helicities. If detector 1 measures helicity ± 1 then detector 2 is guaranteed to measure ∓ 1.

It will be more interesting if the measurements of the photon polarizations are in the linear basis. Then we can look for correlations of the measurement of linear polarization by detector 1 along $\hat{\mathbf{a}}$ and by 2 along $\hat{\mathbf{b}}$. So let's write $| \alpha\rangle$ in the linear polarization basis. The linear and circular polarization bases are related according to

$$| x, y\rangle = \frac{1}{\sqrt{2}}(| 1\rangle \pm i| -1\rangle)$$

Then we can rewrite

$$| \alpha\rangle = \frac{1}{\sqrt{2}} (| x\rangle_1| x\rangle_2 + | y\rangle_1| y\rangle_2)$$

Evidently if detector 1 measures horizontal polarization then so will detector 2, etc. In general we want to measure the correlation $p_{12}(\theta_1, \theta_2)$, that is the probability that we get a count on detetor 1 with polarization axis θ_1 coincident with a count in detector 2 with polarization axis θ_2. The observable is the operator

$$A(\theta_1, \theta_2) = | \theta_1\rangle_1| \theta_2\rangle_2\langle\theta_1 |_1\langle\theta_2 |_2$$

I supppose that we can write

$$A(\theta_1 - \theta_2)$$

Since there is zero angular momentum in the final state, there is rotation symmetry so the observable can only depend on the difference of the polarization angles.

The expectation value of A

$$\begin{aligned} \langle A(\theta_1 - \theta_2)\rangle &= \langle\alpha \mid A \mid \alpha\rangle \\ &= \frac{1}{2} (\langle x |_1\langle x |_2 + \langle y |_1\langle y |_2) \mid \theta_1\rangle_1| \theta_2\rangle_2\langle\theta_1 |_1\langle\theta_2 |_2 (| x\rangle_1| x\rangle_2 + | y\rangle_1| y\rangle_2) \end{aligned}$$

We know that

$$\begin{aligned} \langle x \mid \theta\rangle &= \cos\theta \\ \langle y \mid \theta\rangle &= \sin\theta \end{aligned}$$

with which we can compute

$$\begin{aligned}
\langle A(\theta_1 - \theta_2)\rangle &= \frac{1}{2}\left(\cos^2\theta_1\cos\theta_2^2 + 2\sin\theta_1\cos\theta_1\sin\theta_2\cos\theta_2 + \sin^2\theta_1\sin^2\theta_2\right) \\
&= \frac{1}{2}\left(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\right)^2 \\
&= \frac{1}{2}\left(\cos(\theta_1 - \theta_2)\right)^2 \\
&= \frac{1}{4}\left(1 + \cos 2(\theta_1 - \theta_2)\right)
\end{aligned}$$

The Clauser Horne inequality is

$$\frac{N_{12}(\hat{\mathbf{a}},\hat{\mathbf{b}}) + N_{12}(\hat{\mathbf{b}},\hat{\mathbf{a}}') + N_{12}(\hat{\mathbf{a}}',\hat{\mathbf{b}}') - N_{12}(\hat{\mathbf{a}},\hat{\mathbf{b}}')}{N_1(\hat{\mathbf{a}}') + N_2(\hat{\mathbf{b}})} \le 1$$

If $\hat{\mathbf{a}},\hat{\mathbf{b}},\hat{\mathbf{a}}',\hat{\mathbf{b}}'$ are all separated by the angle ϕ then

$$\frac{N_{12}(\phi) + N_{12}(\phi) + N_{12}(\phi) - N_{12}(3\phi)}{N_1(\hat{\mathbf{a}}') + N_2(\hat{\mathbf{b}})} \le 1$$

$$= \frac{3N_{12}(\phi) - N_{12}(3\phi)}{N_1(\hat{\mathbf{a}}') + N_2(\hat{\mathbf{b}})} \le 1$$

Next relate coincidences to expectation values. $N_{12}(\phi) = N\langle\alpha \mid A(\phi) \mid \alpha\rangle$ As regards the singles counts $N_1(\hat{\mathbf{a}}')$ and $N_2(\hat{\mathbf{b}})$, we know that the number of counts must be independent of the direction of $\hat{\mathbf{a}}'$ or $\hat{\mathbf{b}}$ and that for any direction $N_1 = \frac{1}{2}N$, since half the photons will be polarized along and direction. Therefore the inequality becomes

$$= \frac{(\frac{3}{4}(1 + \cos 2\phi) - \frac{1}{4}(1 + \cos 6\phi)}{\frac{1}{2} + \frac{1}{2}} \le 1$$

$$= \frac{1}{2} + \frac{3}{4}\cos 2\phi - \frac{1}{4}\cos 6\phi \le 1$$

The inequality is maximally violated if $\phi = \pi/8$. Then

$$\frac{1}{2} + \frac{3}{4}\frac{2}{\sqrt{2}} + \frac{1}{4}\frac{2}{\sqrt{2}} \sim 1.2$$

which is not less than 1.

The measurements are consistent with the prediction of quantum mechanics and therefore impossible to reconcile with any kind of local variable theory. If the measurement outcomes are determined before the measurements at detectors 1 or 2 take place, the correlations will satisify the inequality. But the measured correlations do not satisfy the inequality.