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Lecture V

1.1.1 Clauser Horne Inequality

The Bell inequality is difficult to test experimentally. The derivation assumed perfect anticorrelation,namely $C(\hat{\mathbf{n}}, \hat{\mathbf{n}}) = -1$. Clauser and Horne derived an alternative inequality that can be understood in terms of a counting experiment.

Suppose that some observable of the two particles registers as a count in a detector. In the case of the muons we might arrange our Stern Gerlach magnet that is oriented at some angle, so that it kicks spin $+1$ into the detector. If the composite state consists of two photons, the detector registers a hit if the polarization is along some direction. The inequality will be determined by counting. There will be a total of N events, with $N_1(\hat{\mathbf{a}})$ counts in detector 1 when it is set to select $\hat{\mathbf{a}}$ and $N_2(\hat{\mathbf{b}})$ counts in detector 2 when it is set to select $\hat{\mathbf{b}}$. The number of coincidences of the two detectors with settings $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ respectively is $N_{12}(\hat{\mathbf{a}}, \hat{\mathbf{b}})$. The probabilities are

$$p_1(\hat{\mathbf{a}}) = N_1(\hat{\mathbf{a}})/N \quad p_2(\hat{\mathbf{b}}) = N_2(\hat{\mathbf{b}})/N \quad p_{12}(\hat{\mathbf{a}}, \hat{\mathbf{b}})) = N_{12}(\hat{\mathbf{a}}, \hat{\mathbf{b}})/N$$

In this formulation, the hidden variable will determine the probability that $p_1(\hat{\mathbf{a}})$ will have a certain value. Remember that in the Bell formulation, the hidden variable determined absolutely the value of the spin for a particular measurement. Then

$$p_1(\hat{\mathbf{a}}) = \int (p_a(\hat{1}, \lambda) w(\lambda) d\lambda$$

and

$$p_{12}(\hat{\mathbf{a}}, \hat{\mathbf{b}}, \lambda) = p_1(\hat{\mathbf{a}}, \lambda) p_2(\hat{\mathbf{b}}, \lambda)$$

and

$$p_{12}(\hat{\mathbf{a}}, \hat{\mathbf{b}}) = \int p_1(\hat{\mathbf{a}}, \lambda) p_2(\hat{\mathbf{b}}, \lambda) w(\lambda) d\lambda$$

It can be shown that for any four real numbers x, x', y, y' in the range $0 \leq r \leq 1$ that

$$xy - xy' + x'y + x'y' \leq x' + y$$

If we identify the probabilities (all in the range between 0 and 1) as

$$x = p_1(\hat{\mathbf{a}}, \lambda), \quad y = p_2(\hat{\mathbf{b}}, \lambda), \quad x' = p_1(\hat{\mathbf{a}}', \lambda), \quad y' = p_2(\hat{\mathbf{b}}', \lambda)$$

and subsitute into our inequality we get

$$p_1(\hat{\mathbf{a}}, \lambda) p_2(\hat{\mathbf{b}}, \lambda) - p_1(\hat{\mathbf{a}}, \lambda) p_2(\hat{\mathbf{b}}', \lambda) + p_1(\hat{\mathbf{a}}', \lambda) p_2(\hat{\mathbf{b}}, \lambda) + p_1(\hat{\mathbf{a}}', \lambda) p_2(\hat{\mathbf{b}}', \lambda) \leq p_1(\hat{\mathbf{a}}', \lambda) + p_2(\hat{\mathbf{b}}, \lambda)$$

Next multiply by $w(\lambda)$ and integrate over all λ to get

$$p_{12}(\hat{\mathbf{a}}, \hat{\mathbf{b}}) - p_{12}(\hat{\mathbf{a}}, \hat{\mathbf{b}}') + p_{12}(\hat{\mathbf{a}}', \hat{\mathbf{b}}) + p_{12}(\hat{\mathbf{a}}', \hat{\mathbf{b}}') \leq p_1(\hat{\mathbf{a}}') + p_2(\hat{\mathbf{b}})$$

This is the Clauser Horne Inequality. Note that it does not depend on any specific correlations of spin states or perfect anti-correlation.

1.1.2 Experimental Test with 2 photons

An atomic s-state with total angular momentum zero and even parity decays in two steps. Photon γ_1 is emitted in the E1 transition from the S-state to a P-state with $m = \pm 1, 0$. Photon γ_2 is emitted in the second E1 transition to the ground state. The final state of the atom also has zero angular momentum and even parity. Therefore the photons which are emitted back to back have the same helicity, so that their total angular momentum is zero. The two photons have differnt energies, $\omega - 1$ and ω_2. The helicity of each of the photons is determined by the intermediate state. If the intermediate state is $m = +1$ then the helicity of both photons is odd and if $m = -1$ then the helicities are even. The energy of the intermediate state is degenerate. There is no magnetic field that might split the energies of the $m = \pm 1, 0$ levels. The final pure photon state is therefore the linear combination

$$|\,\alpha\rangle = \frac{1}{\sqrt{2}}\left(|\,+1\rangle|\,+1\rangle + |\,-1\rangle|\,-1\rangle\right)$$

where the states $|\,\pm 1\rangle$ are the right and left handed helicities. If detector 1 measures helicity ± 1 then detector 2 is guaranteed to measure ∓ 1.

It will be more interesting if the measurements of the photon polarizations are in the linear basis. Then we can look for correlations of the measurement of linear polarization by detector 1 along $\hat{\mathbf{a}}$ and by 2 along $\hat{\mathbf{b}}$. So let's write $|\,\alpha\rangle$ in the linear polarization basis. The linear and circular polarization bases are related according to

$$|\,x, y, k\rangle = \frac{1}{\sqrt{2}}(|\,1\rangle \pm i|\,-1\rangle)$$

and

$$|\,x, y, -k\rangle = \frac{1}{\sqrt{2}}(|\,1\rangle \mp i|\,-1\rangle)$$

Then we can rewrite

$$|\,\alpha\rangle = \frac{1}{\sqrt{2}}\left(|\,x\rangle_1|\,x\rangle_2 + |\,y\rangle_1|\,y\rangle_2\right)$$

Evidently if detector 1 measures horizontal polarization then so will detector 2, etc. In general we want to measure the correlation $p_{12}(\theta_1, \theta_2)$, that is the probability that we get a count on detetor 1 with polarization axis θ_1 coincident with a count in detector 2 with polarization axis θ_2. The observable is the operator

$$A(\theta_1, \theta_2) = |\,\theta_1\rangle_1|\,\theta_2\rangle_2\langle\theta_1\,|_1\langle\theta_2\,|_2$$

I supppose that we can write

$$A(\theta_1 - \theta_2)$$

Since there is zero angular momentum in the final state, there is rotation symmetry so the observable can only depend on the difference of the polarization angles.

The expectation value of A

$$\begin{aligned} \langle A(\theta_1 - \theta_2)\rangle &= \langle\alpha\mid A\mid\alpha\rangle \\ &= \frac{1}{2}\left(\left(\langle x\,|_1\langle x\,|_2 + \langle y\,|_1\langle y\,|_2\right)|\,\theta_1\rangle_1|\,\theta_2\rangle_2\langle\theta_1\,|_1\langle\theta_2\,|_2\left(|\,x\rangle_1|\,x\rangle_2 + |\,y\rangle_1|\,y\rangle_2\right)\right) \end{aligned}$$

We know that

$$\begin{aligned}\langle x \mid \theta\rangle &= \cos\theta \\ \langle y \mid \theta\rangle &= \sin\theta\end{aligned}$$

with which we can compute

$$\begin{aligned}\langle A(\theta_1-\theta_2)\rangle &= \frac{1}{2}\left(\cos^2\theta_1\cos\theta_2^2 + 2\sin\theta_1\cos\theta_1\sin\theta_2\cos\theta_2 + \sin^2\theta_1\sin^2\theta_2\right) \\ &= \frac{1}{2}\left(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\right)^2 \\ &= \frac{1}{2}\left(\cos(\theta_1-\theta_2)\right)^2 \\ &= \frac{1}{4}\left(1+\cos 2(\theta_1-\theta_2)\right)\end{aligned}$$

The Clauser Horne inequality is

$$\frac{N_{12}(\hat{\mathbf{a}},\hat{\mathbf{b}}) + N_{12}(\hat{\mathbf{b}},\hat{\mathbf{a}}') + N_{12}(\hat{\mathbf{a}}',\hat{\mathbf{b}}') - N_{12}(\hat{\mathbf{a}},\hat{\mathbf{b}}')}{N_1(\hat{\mathbf{a}}') + N_2(\hat{\mathbf{b}})} \leq 1$$

If $\hat{\mathbf{a}},\hat{\mathbf{b}},\hat{\mathbf{a}}',\hat{\mathbf{b}}'$ are all separated by the angle ϕ then

$$\frac{N_{12}(\phi)+N_{12}(\phi)+N_{12}(\phi)-N_{12}(3\phi)}{N_1(\hat{\mathbf{a}}')+N_2(\hat{\mathbf{b}})} \leq 1$$

$$= \frac{3N_{12}(\phi)-N_{12}(3\phi)}{N_1(\hat{\mathbf{a}}')+N_2(\hat{\mathbf{b}})} \leq 1$$

Next relate coincidences to expectation values. $N_{12}(\phi) = N\langle\alpha \mid A(\phi) \mid \alpha\rangle$ As regards the singles counts $N_1(\hat{\mathbf{a}}')$ and $N_2(\hat{\mathbf{b}})$, we know that the number of counts must be independent of the direction of $\hat{\mathbf{a}}'$ or $\hat{\mathbf{b}}$ and that for any direction $N_1 = \frac{1}{2}N$, since half the photons will be polarized along and direction. Therefore the inequality becomes

$$= \frac{(\frac{3}{4}(1+\cos 2\phi) - \frac{1}{4}(1+\cos 6\phi)}{\frac{1}{2}+\frac{1}{2}} \leq 1$$

$$= \frac{1}{2} + \frac{3}{4}\cos 2\phi - \frac{1}{4}\cos 6\phi \leq 1$$

The inequality is maximally violated if $\phi = \pi/8$. Then

$$\frac{1}{2} + \frac{3}{4}\frac{2}{\sqrt{2}} + \frac{1}{4}\frac{2}{\sqrt{2}} \sim 1.2$$

which is not less than 1.

The measurements are consistent with the prediction of quantum mechanics and therefore impossible to reconcile with any kind of local variable theory. If the measurement outcomes are determined before the measurements at detectors 1 or 2 take place, the correlations will satisify the inequality. But the measured correlations do not satisfy the inequality.

1.2 No Clone Theorem

Suppose there is a unitary transformation that copies a state $|\,\psi\rangle$ to a state subject to some standard initialization.

$$U|\,\psi\rangle|\,0\rangle = |\,\psi\rangle|\,\psi\rangle$$

We will find a contradiction if we follow the logic. Let $|\,\psi\rangle = \alpha|\,1\rangle + \beta|\,0\rangle$. Then

$$U|\,\psi\rangle|\,0\rangle = U\left(\alpha|\,1\rangle + \beta|\,0\rangle\right)|\,0\rangle = \alpha|\,1\rangle|\,1\rangle + \beta|\,0\rangle|\,0\rangle.$$

But we showed above that

$$U|\,\psi\rangle|\,0\rangle = |\,\psi\rangle|\,\psi\rangle = \left(\alpha|\,1\rangle + \beta|\,0\rangle\right)\left(\alpha|\,1\rangle + \beta|\,0\rangle\right) = \alpha^2|\,1\rangle|\,1\rangle + \alpha\beta|\,1\rangle[|\,0\rangle + \beta\alpha|\,0\rangle|\,1\rangle + \beta^2|\,0\rangle|\,0\rangle$$

1.3 Teleportation

Alice has a quantum state, a single q-bit. And she wants to send that state to Bob. Unfortunately she does not know what the state is

$$| \alpha\rangle = a| 0\rangle + \beta| 1\rangle.$$

If she did she could just tell Bob over a telephone line and he could make his own version. Also, she only has one of these states. She would need an ensemble of copies to determine the state from measurement.

Suppose they share the components of an entangled state. We consider the standard maximally entangled 2 qubit state

$$| \psi\rangle = \frac{1}{\sqrt{2}}[| 1\rangle| 1\rangle + | 0\rangle| 0\rangle]$$

Bob can measure his bit and Alice hers. Of course Alice's measurement determines Bob's. Now Alice wants to teleport her unknown state $| \alpha\rangle$. She puts it together with the entangled state

$$\frac{1}{\sqrt{2}}(a| 0\rangle + \beta| 1\rangle)[| 1\rangle| 1\rangle + | 0\rangle| 0\rangle]$$

or

$$\frac{1}{\sqrt{2}}[\alpha| 0\rangle| 1\rangle| 1\rangle + \beta| 1\rangle| 1\rangle| 1\rangle + \alpha| 0\rangle| 0\rangle| 0\rangle + \beta| 1\rangle| 0\rangle| 0\rangle]$$

$$= \frac{1}{2}\left(| \Phi^+\rangle(a| 0\rangle + b| 1\rangle) + | \Phi^-\rangle(a| 0\rangle - b| 1\rangle) + | \Psi^+\rangle(a| 1\rangle + b| 0\rangle) + | \Psi^+\rangle(a| 1\rangle - b| 0\rangle)\right)$$

where

$$\begin{aligned}
| \Phi^+\rangle &= \frac{1}{\sqrt{2}}(| 00\rangle + | 11\rangle) \\
| \Phi^-\rangle &= \frac{1}{\sqrt{2}}(| 00\rangle - | 11\rangle) \\
| \Psi^+\rangle &= \frac{1}{\sqrt{2}}(| 01\rangle + | 10\rangle) \\
| \Psi^-\rangle &= \frac{1}{\sqrt{2}}(| 01\rangle - | 10\rangle)
\end{aligned}$$

Alice has access to the first two q-bits, and Bob to the third. Alice then performs a measurement on her two q-bits. She transforms to the Bell basis $| \Phi^+\rangle, | \Phi^-\rangle, | \Psi^+\rangle, | \Psi^-\rangle$ and the outcome of her measurement is that the system is in one of the four eigenstates. She then calls Bob and tells him the eigenvalue. (the state of her half of the entangled state is know to her.)

If she measures the system to be in the state

- $| \Phi^+\rangle$ - Then Bob's q-bit is in the state $a| 0\rangle + b| 1\rangle$ and she has managed to transport her state to Bob
- $| \Phi^-\rangle$ - Then Bob's q-bit is in the state $a| 0\rangle - b| 1\rangle$ and she has managed to transport her state to Bob. He then performs the unitary transformation $\sigma_z(a| 0\rangle - b| 1\rangle) = | \alpha\rangle$

- $| \Psi^+\rangle$ - Then Bob's q-bit is in the state $a| 1\rangle + b| 0\rangle$ and she has managed to transport her state to Bob. He then performs the unitary transformation $\sigma_x(a| 1\rangle + b| 0\rangle) = | \alpha\rangle$
- $| \Psi^-\rangle$ - Then Bob's q-bit is in the state $a| 1\rangle - b| 0\rangle$ and she has managed to transport her state to Bob. He then performs the unitary transformation $\sigma_z\sigma_x(a| 1\rangle - b| 0\rangle) = | \alpha\rangle$

Of course the protocol was established in advance. Bob now has his hands on the state that Alice started with. Alice is left without the state. (Otherwise we would have a copy) If she attempts a measurement of the unkown Q-bit she will get 0 or 1 with equal probability. That is, it is maximally mixed.

Teleportation requres that the state that Alice and Bob share is entangled. Otherwise the shared state looks like

$$| 0\rangle| 0\rangle,$$

(again Bob can see the Q-bit to the right and Alice to the left.) Together with the state to be transported we have

$$| \alpha\rangle| 0\rangle| 0\rangle$$

No matter what Alice does to the states at her disposal, Bob's state is simply $| 0\rangle$.