February 2, 2015
Lecture VI

1.2 Deutsch's Problem

Do the non-local correlations peculiar to quantum mechanics provide a computational advantage. Consider Deutsch's problem. Suppose we have a simple system of two bits, an input register and output register.

$$\mid x\rangle \mid y\rangle$$

Each bit can have the value of 0 or 1. Define the operation of the computer as

$$U_f : \mid x\rangle \mid y\rangle = \mid x\rangle \mid y \oplus f(x)\rangle$$

where $f(x)$ can have the value 0 or 1 and $y \oplus f(x) = 0$ if $y = f(x)$ or 1 if $y \neq f(x)$. In order to learn what $f(x)$ does we need to run the machinery that does the calculation twice, to determine $f(0)$ and then $f(1)$. But perhaps we are not interested in learning the values of $f(0)$ and $f(1)$ but rather whether $f(0) = f(1)$ (the function is constant) or $f(0) \neq f(1)$ (the function is balanced). If our computer is classical we still need to run it twice to evaluate $f(0)$ and $f(1)$. But if it is quantum mechanical we have options.

First let's initiate the output register in the state

$$\mid y\rangle = \frac{1}{\sqrt{2}}[\mid 0\rangle - \mid 1\rangle]$$

Then

$$\begin{aligned} U_f : \mid x\rangle \mid y\rangle &= U_f : \mid x\rangle \frac{1}{\sqrt{2}}[\mid 0\rangle - \mid 1\rangle] \\ &= \mid x\rangle \frac{1}{\sqrt{2}}[\mid 0 + f(x)\rangle - \mid 1 + f(x)\rangle] \\ &= \begin{array}{ll} \mid x\rangle \frac{1}{\sqrt{2}}[\mid 0\rangle - \mid 1\rangle] & \text{if } f(x) = 0 \\ \mid x\rangle \frac{1}{\sqrt{2}}[\mid 1\rangle - \mid 0\rangle] & \text{if } f(x) = 1 \end{array} \end{aligned} \tag{1.1}$$

or

$$U_f : \mid x\rangle \mid y\rangle = U_f : \mid x\rangle (-1)^{f(x)} \frac{1}{\sqrt{2}}[\mid 0\rangle - \mid 1\rangle]$$

The next step is to initialize the input register. Let

$$\mid x\rangle = \frac{1}{\sqrt{2}}[\mid 0\rangle + \mid 1\rangle]$$

Now

$$\begin{aligned} U_f : \mid x\rangle \mid y\rangle &= U_f : \frac{1}{\sqrt{2}}[\mid 0\rangle + \mid 1\rangle] \frac{1}{\sqrt{2}}[\mid 0\rangle - \mid 1\rangle] \\ &= \frac{1}{\sqrt{2}}[\mid 0\rangle(-1)^{f(0)} + \mid 1\rangle(-1)^{f(1)}] \frac{1}{\sqrt{2}}[\mid 0\rangle - \mid 1\rangle] \end{aligned}$$

If $f(0) = f(1)$ then

$$U_f : | x\rangle | y\rangle = \frac{1}{\sqrt{2}}(-1)^{f(0)}[| 0\rangle + | 1\rangle]\frac{1}{\sqrt{2}}[| 0\rangle - | 1\rangle]$$

If $f(0) \neq f(1)$ then

$$U_f : | x\rangle | y\rangle = \frac{1}{\sqrt{2}}(-1)^{f(0)}[| 0\rangle - | 1\rangle]\frac{1}{\sqrt{2}}[| 0\rangle - | 1\rangle]$$

Finally make a measurement in the basis

$$| \pm\rangle = \frac{1}{\sqrt{2}}[| 0\rangle \pm | 1\rangle]$$

If the input register is in the state $| +\rangle$ then $f(0) = f(1)$ and if it is in the state $| -\rangle$, $f(0) \neq f(1)$. That the function is balanced or constant is determined with a single operation of the computer. This is not possible with a classical computer. It is an example of quantum parallelism.

1.3 Introduction

Scattering is perhaps the most important experimental technique for exploring the structure of matter.

- From Rutherford's measurement that informed the "planetary" in place of the plum pudding model, of negative charges circulating about a compact positively charged nucleus. Of course the energy of the alpha particle probe needed to be sufficient to penetrate the electron coud so that it would see the full nuclear charge.
- Scattering of electrons from nuclei revealed the structure of first of the nuclei (protons and neutrons) and then fractionally charged quarks, quark spins, quark momentum distriburtions within the proton.
- Neutrino, nuclei scattering experiments demonstratd the weak coupling of quarks, via W and Z_0 bosons.
- Electron-positron scattering lead to a new field of heavyy quark spectroscopy, the study of the bound states of charm and beauty quarks and anti-quarks.
- Total e^+e^- cross section for hadron production was direct observation of color degree of freedom, that quards come in 3 colors
- Forward backward asymmetry in e^+e^- scattering revealed the weak neutral current coupling of quarks as well as leptons.
- The ultimate scattering experiments at LHC have unearthed a Higgs, and perhaps ?
- xray scattering is used to determine the structure of crystals.
- And x-ray and low energy electron scattering and neutron scattering is the principle tool for the study of condensed matter.

- Parity violation in weak interactions, CP violation in Kaon and B-meson decay, etc.

We have a nice neat picture today, the Standar Model. The experimantal evidence is the product of mostly scattering experiments.

1.4 Lippman-Schwinger Equation

We start with the simplest description of scattering, that is, a free particle (plane wave) interacting with a fixed and localized potential. Localized means that the potential falls off rapidly far from the origin. And plane wave meaning a state with definite energy and momentum. While scattering is inherently time dependent we begin by thinking about a time independent version, perhaps a continuous stream of incident and then scattered particles. As this is quantum mechnaics, the Schrodinger equation will play a big role. We will represent the scattering particle as a solution to SE. But what do we measure? Not a wave function. We measure a flux of scattered particles at some angle, energy, spin? So we need a way to translate from wave function (solution to SE) to flux.

Of course if we think about classical scattering, then we have an impact parameter. We solve the equations of motion, and relate impact parameter to scattering angle. But that does not work for QM.

The strategy is first to find a solution to the time independent Schrodinger equation for free particle states, where that solution is a plane wave from the scattering center. Picture a plane wave from $-\infty$ at the left to $+\infty$ to the right and a spherical wave centered around the origin of the potential. We want the solution for a particular energy, namely the energy of the incident particle. We start assuming elastic scattering. The energy of the scattered particle is fixed.

We start with the Schrodinger equation

$$\begin{aligned} (H_0+V)|\,\psi\rangle &= E|\,\psi\rangle \\ \rightarrow (E-H_0)|\,\psi\rangle &= V|\,\psi\rangle \end{aligned}$$

where H_0 is the kinetic energy of the free particle and V the scattering potential. Again, we imagine an incoming plane wave, interaction with some potential, and then time independent solution. We "solve" as follows

$$\rightarrow |\,\psi\rangle = \frac{1}{E-H_0}V|\,\psi\rangle + |\,\phi\rangle \tag{1.2}$$

where $H_0|\,\phi\rangle = E|\,\phi\rangle$. That way when $V\rightarrow 0$, $|\,\psi\rangle \rightarrow |\,\phi\rangle$. Also, far from the scattering center, $|\,\psi\rangle \rightarrow |\,\phi\rangle$. So far we have an operator equation. We can translate it to some concrete algebra by inserting some basis states. But first we note that H_0 has a continuous spectrum that will include E, so we replace E with the complex variable $E \pm i\epsilon$. Then

$$\rightarrow |\,\psi\rangle = \frac{1}{E-H_0\pm i\epsilon}V|\,\psi\rangle + |\,\phi\rangle \tag{1.3}$$

In the coordinate basis

$$\begin{aligned} \langle x\mid\psi\rangle &= \langle x\mid\frac{1}{E-H_0\pm i\epsilon}V|\,\psi\rangle + \langle x\mid\phi\rangle \\ &= \int\int d^3x''d^3x'\langle x\mid\frac{1}{E-H_0\pm i\epsilon}|\,x'\rangle\langle x'\,|V|\,x''\rangle\langle x''\mid\psi\rangle + \langle x\mid\phi\rangle \end{aligned}$$

For a local potential $\langle x' \mid V \mid x'' \rangle = \delta^3(x' - x'')V(x')$

$$\begin{aligned}
\langle x \mid \psi \rangle &= \langle x \mid \frac{1}{E - H_0 \pm i\epsilon} V \mid \psi \rangle + \langle x \mid \phi \rangle \\
&= \int d^3x' \langle x \mid \frac{1}{E - H_0 \pm i\epsilon} \mid x' \rangle \langle x' \mid V \mid x' \rangle \langle x' \mid \psi \rangle + \langle x \mid \phi \rangle
\end{aligned}$$

Let's try to evaluate

$$\begin{aligned}
G(x, x') &= \frac{\hbar^2}{2m} \langle x \mid \frac{1}{E - H_0 \pm i\epsilon} \mid x' \rangle \\
&= \frac{\hbar^2}{2m} \int d^3p'' d^3p' \langle x \mid p' \rangle \langle p' \mid \frac{1}{E - H_0 \pm i\epsilon} \mid p'' \rangle \langle p'' \mid x' \rangle \\
&= \hbar^2 \int d^3p' \langle x \mid p' \rangle \langle p' \mid p'' \rangle \frac{1}{p^2 - {p'}^2 \pm i\epsilon} \langle p'' \mid x' \rangle \\
&= \frac{1}{(2\pi)^3 \hbar} \int d^3p' \frac{e^{i(x-x')\cdot p'/\hbar}}{p^2 - {p'}^2 \pm i\epsilon} \quad \text{where we use } \left\langle p \mid p'' \right\rangle = \delta^3(\mathbf{p}' - \mathbf{p}''), \text{ and } \langle p \mid x \rangle = \frac{e^{-ix\cdot p/\hbar}}{(2\pi\hbar)^{3/2}} \\
&= \frac{1}{(2\pi)^3 \hbar} \int d^3p' \frac{e^{i|x-x'|p' \cos\theta/\hbar}}{p^2 - {p'}^2 \pm i\epsilon}
\end{aligned}$$

Next integrate around θ

$$\begin{aligned}
G(x, x') &= \frac{1}{(2\pi)^3 \hbar} 2\pi \int_0^\infty {p'}^2 dp' \int d\phi d(\cos\theta) \frac{e^{i|x-x'|p' \cos\theta/\hbar}}{p^2 - {p'}^2 \pm i\epsilon} \\
&= -\frac{1}{(2\pi)^3 \hbar} \frac{2\pi}{i|x - x'|} \int_0^\infty \frac{\hbar}{p'} {p'}^2 dp' \frac{e^{i|x-x'|p'/\hbar} - e^{-i|x-x'|p'/\hbar}}{p^2 - {p'}^2 \pm i\epsilon} \\
&= -\frac{1}{8\pi^2} \frac{1}{i|x - x'|} \int_{-\infty}^\infty k' dk' \frac{e^{i|x-x'|k'} - e^{-i|x-x'|k'}}{k^2 - {k'}^2 \pm i\epsilon} \\
&= -\frac{1}{8\pi^2} \frac{1}{i|x - x'|} \int_{-\infty}^\infty k' dk' \frac{e^{i|x-x'|k'} - e^{-i|x-x'|k'}}{(k - k' \pm i\epsilon)(k + k' \pm i\epsilon)}
\end{aligned}$$

Next do the contour integral. We first consider the contribution from the denominator $E - H_0 + i\epsilon$. This can be written in terms of k where $E = \hbar^2 k^2/2m$ and k' as $(k + i\epsilon' - k')(k + i\epsilon' + k')$ With that in mind let's rewrite our last expression just for the $+i\epsilon$ piece. Then

$$G(x, x') = -\frac{1}{8\pi^2} \frac{1}{i|x - x'|} \int_{-\infty}^\infty k' dk' \frac{e^{i|x-x'|k'} - e^{-i|x-x'|k'}}{(k + i\epsilon' - k')(k + i\epsilon' + k')}$$

There are poles at $k' = k + i\epsilon'$ and $k' = -k - i\epsilon'$. (The poles for denominator $E - H_0 - i\epsilon$ are shown in Figure 1.2.) For the positive exponent we close in the upper half plane and include the pole at $k' = k + i\epsilon$. For the negative exponent close in the lower half plane and include the pole at $k' = -k - i\epsilon$. Each term contributes

$$2\pi i k \frac{e^{i|x-x'|k}}{2k}$$

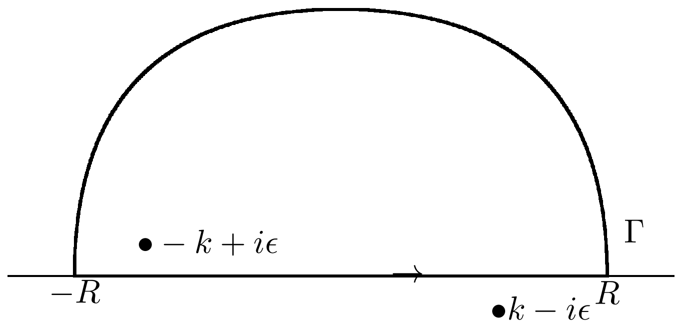

Figure 1.1: Poles for denominator $E - H_0 - i\epsilon$.

Again for $+\imath\epsilon$ we get

$$\begin{aligned} G(x,x') &= -\frac{1}{8\pi^2}\frac{1}{i|x-x'|}\int_{-\infty}^{\infty} k'dk' \frac{e^{i|x-x'|k'} - e^{-i|x-x'|k'}}{(k+i\epsilon'-k')(k+i\epsilon'+k')} \\ &= -\frac{1}{8\pi^2}\frac{2\pi i}{i|x-x'|}e^{i|x-x'|k} \\ &= -\frac{1}{4\pi}\frac{1}{|x-x'|}e^{i|x-x'|k} \end{aligned}$$

For $E - H_0 = i\epsilon$, we would get

$$G(x,x') = -\frac{1}{4\pi}\frac{1}{|x-x'|}e^{i|x-x'|k} \tag{1.4}$$

We are interested in the solution far from the scattering origin, since that is where we will detect the scattered particle. In fact, the scattered particle will be a plane wave with definite momentum, just like the incident particle, but in a different direction. So let's inspect G in the limit where $|x - x'|$ is large, and $|x|$ is much greater than the range of the potential.

$$\begin{aligned} G(x,x') &= -\frac{1}{4\pi|x-x'|}e^{i|x-x'|p/\hbar} \sim -\frac{e^{i(x^2+x'^2-2x\cdot x')^{1/2}p/\hbar}}{4\pi r} \\ &\sim -\frac{e^{ir(1-x\cdot x'/r^2)p/\hbar}}{4\pi r} \\ &\sim -\frac{e^{irp/\hbar}e^{-i(\hat{x}\cdot x')p/\hbar}}{4\pi r} \end{aligned}$$

Now if $k' = \hat{x}|p|/\hbar$, that is k' is set to be in the outgoing x direction to the detector.

$$\rightarrow_{x\gg x'} \quad = \quad -\frac{e^{ikr}}{4\pi r}e^{-ix'\cdot k'}$$

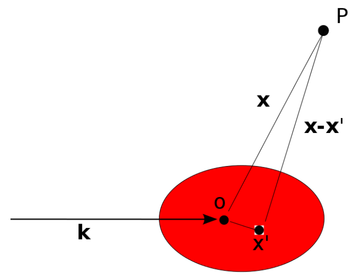

Figure 1.2: Red is the region of the scattering potential. The incident plane wave is headed in the **k** direction. The scattered particle is observed at P. The direction of propagation of the outgoing plane wave is **x**.

Finally we can write

$$\begin{aligned}
\langle x \mid \psi\rangle &= \frac{2m}{\hbar^2}\int d^3x' G(x,x')V(x')\langle x' \mid \psi\rangle + \langle x \mid k\rangle \\
\langle x \mid \psi\rangle &= -\frac{2m}{\hbar^2}\frac{e^{ikr}}{4\pi r}\int d^3x' e^{-ix'\cdot\hat{x}p/\hbar}V(x')\langle x' \mid \psi\rangle + \langle x \mid k\rangle \\
\langle x \mid \psi\rangle &= -\frac{2m}{\hbar^2}\frac{e^{ikr}}{4\pi r}\int d^3x' e^{-ix'\cdot k'}V(x')\langle x' \mid \psi\rangle + \langle x \mid k\rangle \\
&= -\frac{2m}{\hbar^2}\frac{e^{ikr}}{4\pi r}\int d^3x' e^{-ix'\cdot k'}V(x')\langle x' \mid \psi\rangle + \frac{e^{ix\cdot k}}{(2\pi)^{3/2}} \\
&= \frac{1}{(2\pi)^{3/2}}\left(-\frac{e^{ikr}}{r}\frac{2m(2\pi)^3}{4\pi\hbar^2}\int d^3x' \frac{e^{-ix'\cdot k'}}{(2\pi)^{3/2}}V(x')\langle x' \mid \psi\rangle + e^{ix\cdot k}\right)
\end{aligned}$$

Define the scattering amplitude

$$f(k',k) = -\frac{4\pi^2 m}{\hbar^2}\int d^3x'\langle k' \mid x'\rangle V(x')\langle x' \mid \psi^+\rangle \tag{1.5}$$

Then

$$\langle x \mid \psi\rangle \sim \frac{1}{(2\pi)^{3/2}}\left(\frac{e^{ikr}}{r}f(k',k) + e^{ik\cdot x}\right)$$