February 4, 2015
Lecture VII

1.2 Scattering

Scattering is perhaps the most important experimental technique for exploring the structure of matter.

- From Rutherford's measurement that informed the "planetary" in place of the plum pudding model, of negative charges circulating about a compact positively charged nucleus. Of course the energy of the alpha particle probe needed to be sufficient to penetrate the electron coud so that it would see the full nuclear charge.
- Scattering of electrons from nuclei revealed the structure of first of the nuclei (protons and neutrons) and then fractionally charged quarks, quark spins, quark momentum distriburtions within the proton.
- Neutrino, nuclei scattering experiments demonstratd the weak coupling of quarks, via W and Z_0 bosons.
- Electron-positron scattering lead to a new field of heavyy quark spectroscopy, the study of the bound states of charm and beauty quarks and anti-quarks.
- Total e^+e^- cross section for hadron production was direct observation of color degree of freedom, that quards come in 3 colors
- Forward backward asymmetry in e^+e^- scattering revealed the weak neutral current coupling of quarks as well as leptons.
- The ultimate scattering experiments at LHC have unearthed a Higgs, and perhaps ?
- xray scattering is used to determine the structure of crystals.
- And x-ray and low energy electron scattering and neutron scattering is the principle tool for the study of condensed matter.
- Parity violation in weak interactions, CP violation in Kaon and B-meson decay, etc.

We have a nice neat picture today, the Standar Model. The experimantal evidence is the product of mostly scattering experiments.

1.3 Lippman-Schwinger Equation

We start with the simplest description of scattering, that is, a free particle (plane wave) interacting with a fixed and localized potential. Localized means that the potential falls off rapidly far from the origin. And plane wave meaning a state with definite energy and momentum. While scattering is inherently time dependent we begin by thinking about a time independent version, perhaps a continuous stream of incident and then scattered particles. As this is quantum mechnaics, the Schrodinger equation will play a big role. We will represent the scattering particle as a solution to

SE. But what do we measure? Not a wave function. We measure a flux of scattered particles at some angle, energy, spin? So we need a way to translate from wave function (solution to SE) to flux.

Of course if we think about classical scattering, then we have an impact parameter. We solve the equations of motion, and relate impact parameter to scattering angle. But that does not work for QM.

The strategy is first to find a solution to the time independent Schrodinger equation for free particle states, where that solution is a plane wave from the scattering center. Picture a plane wave from $-\infty$ at the left to $+\infty$ to the right and a spherical wave centered around the origin of the potential. We want the solution for a particular energy, namely the energy of the incident particle. We start assuming elastic scattering. The energy of the scattered particle is fixed.

We start with the Schrodinger equation

$$\begin{aligned} (H_0 + V)|\,\psi\rangle &= E|\,\psi\rangle \\ \rightarrow (E - H_0)|\,\psi\rangle &= V|\,\psi\rangle \end{aligned}$$

where H_0 is the kinetic energy of the free particle and V the scattering potential. Again, we imagine an incoming plane wave, interaction with some potential, and then time independent solution. We "solve" as follows

$$\rightarrow |\,\psi\rangle = \frac{1}{E - H_0} V|\,\psi\rangle + |\,\phi\rangle \tag{1.1}$$

where $H_0|\,\phi\rangle = E|\,\phi\rangle$. That way when $V \rightarrow 0$, $|\,\psi\rangle \rightarrow |\,\phi\rangle$. Also, far from the scattering center, $|\,\psi\rangle \rightarrow |\,\phi\rangle$. So far we have an operator equation. We can translate it to some concrete algebra by inserting some basis states. But first we note that H_0 has a continuous spectrum that will include E, so we replace E with the complex variable $E \pm i\epsilon$. Then

$$\rightarrow |\,\psi\rangle = \frac{1}{E - H_0 \pm i\epsilon} V|\,\psi\rangle + |\,\phi\rangle \tag{1.2}$$

In the coordinate basis

$$\begin{aligned} \langle x \mid \psi\rangle &= \langle x \,|\frac{1}{E - H_0 \pm i\epsilon} V|\,\psi\rangle + \langle x \mid \phi\rangle \\ &= \int\int d^3x'' d^3x' \langle x\,|\frac{1}{E - H_0 \pm i\epsilon}|\,x'\rangle\langle x'\,|V|\,x''\rangle\langle x'' \mid \psi\rangle + \langle x \mid \phi\rangle \end{aligned}$$

For a local potential $\langle x' \mid V \mid x''\rangle = \delta^3(x' - x'')V(x')$

$$\begin{aligned} \langle x \mid \psi\rangle &= \langle x \,|\frac{1}{E - H_0 \pm i\epsilon} V|\,\psi\rangle + \langle x \mid \phi\rangle \\ &= \int d^3x' \langle x\,|\frac{1}{E - H_0 \pm i\epsilon}|\,x'\rangle\langle x'\,|V|\,x'\rangle\langle x' \mid \psi\rangle + \langle x \mid \phi\rangle \end{aligned}$$

Let's try to evaluate

$$G(x, x') = \frac{\hbar^2}{2m}\langle x\,|\frac{1}{E - H_0 \pm i\epsilon}|\,x'\rangle$$

$$
\begin{aligned}
&= \frac{\hbar^2}{2m}\int d^3p'' d^3p' \langle x \mid p'\rangle\langle p' \mid \frac{1}{E-H_0 \pm i\epsilon} \mid p''\rangle\langle p'' \mid x'\rangle \\
&= \hbar^2 \int d^3p' \langle x \mid p'\rangle\langle p' \mid p''\rangle \frac{1}{p^2 - {p'}^2 \pm i\epsilon}\langle p'' \mid x'\rangle \\
&= \frac{1}{(2\pi)^3\hbar}\int d^3p' \frac{e^{i(x-x')\cdot p'/\hbar}}{p^2-{p'}^2 \pm i\epsilon} \quad \text{where we use } \left\langle p \mid p'' \right\rangle = \delta^3(\mathbf{p}'-\mathbf{p}''), \text{ and } \langle p \mid x\rangle = \frac{e^{-ix\cdot p/\hbar}}{(2\pi\hbar)^{3/2}} \\
&= \frac{1}{(2\pi)^3\hbar}\int d^3p' \frac{e^{i|x-x'|p'\cos\theta/\hbar}}{p^2-{p'}^2 \pm i\epsilon}
\end{aligned}
$$

Next integrate around θ

$$
\begin{aligned}
G(x,x') &= \frac{1}{(2\pi)^3\hbar}2\pi\int_0^\infty {p'}^2 dp' \int d\phi d(\cos\theta)\frac{e^{i|x-x'|p'\cos\theta/\hbar}}{p^2-{p'}^2 \pm i\epsilon} \\
&= -\frac{1}{(2\pi)^3\hbar}\frac{2\pi}{i|x-x'|}\int_0^\infty \frac{\hbar}{p'}{p'}^2 dp' \frac{e^{i|x-x'|p'/\hbar}-e^{-i|x-x'|p'/\hbar}}{p^2-{p'}^2 \pm i\epsilon} \\
&= -\frac{1}{8\pi^2}\frac{1}{i|x-x'|}\int_{-\infty}^\infty k'dk' \frac{e^{i|x-x'|k'}-e^{-i|x-x'|k'}}{k^2-{k'}^2 \pm i\epsilon} \\
&= -\frac{1}{8\pi^2}\frac{1}{i|x-x'|}\int_{-\infty}^\infty k'dk' \frac{e^{i|x-x'|k'}-e^{-i|x-x'|k'}}{(k-k' \pm i\epsilon)(k+k' \pm i\epsilon)}
\end{aligned}
$$

Next do the contour integral. We first consider the contribution from the denominator $E-H_0+i\epsilon$. This can be written in terms of k where $E=\hbar^2k^2/2m$ and k' as $(k+i\epsilon'-k')(k+i\epsilon'+k')$ With that in mind let's rewrite our last expression just for the $+i\epsilon$ piece. Then

$$
G(x,x') = -\frac{1}{8\pi^2}\frac{1}{i|x-x'|}\int_{-\infty}^\infty k'dk' \frac{e^{i|x-x'|k'}-e^{-i|x-x'|k'}}{(k+i\epsilon'-k')(k+i\epsilon'+k')}
$$

There are poles at $k'=k+i\epsilon'$ and $k'=-k-i\epsilon'$. (The poles for denominator $E-H_0-i\epsilon$ are shown in Figure 1.2.) For the positive exponent we close in the upper half plane and include the pole at $k'=k+i\epsilon$. For the negative exponent close in the lower half plane and include the pole at $k'=-k-i\epsilon$. Each term contributes

$$
2\pi ik\frac{e^{i|x-x'|k}}{2k}
$$

Again for $+\imath\epsilon$ we get

$$
\begin{aligned}
G(x,x') &= -\frac{1}{8\pi^2}\frac{1}{i|x-x'|}\int_{-\infty}^\infty k'dk' \frac{e^{i|x-x'|k'}-e^{-i|x-x'|k'}}{(k+i\epsilon'-k')(k+i\epsilon'+k')} \\
&= -\frac{1}{8\pi^2}\frac{2\pi i}{i|x-x'|}e^{i|x-x'|k} \\
&= -\frac{1}{4\pi}\frac{1}{|x-x'|}e^{i|x-x'|k}
\end{aligned}
$$

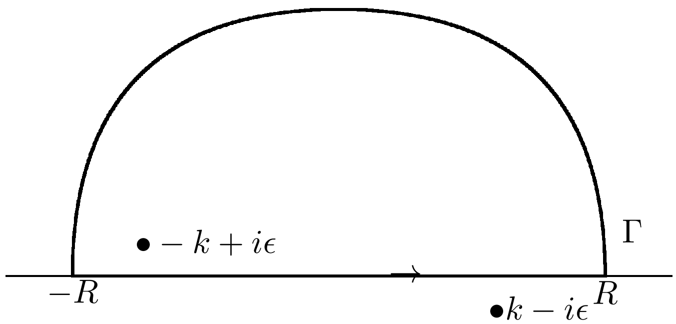

Figure 1.1: Poles for denominator $E - H_0 - i\epsilon$.

For $E - H_0 = i\epsilon$, we would get

$$G(x, x') = -\frac{1}{4\pi}\frac{1}{|x - x'|}e^{i|x-x'|k} \tag{1.3}$$

We are interested in the solution far from the scattering origin, since that is where we will detect the scattered particle. In fact, the scattered particle will be a plane wave with definite momentum, just like the incident particle, but in a different direction. So let's inspect G in the limit where $|x - x'|$ is large, and $|x|$ is much greater than the range of the potential.

$$\begin{aligned} G(x, x') &= -\frac{1}{4\pi|x - x'|}e^{i|x-x'|p/\hbar} \sim -\frac{e^{i(x^2+x'^2-2x\cdot x')^{1/2}p/\hbar}}{4\pi r} \\ &\sim -\frac{e^{ir(1-x\cdot x'/r^2)p/\hbar}}{4\pi r} \\ &\sim -\frac{e^{irp/\hbar}e^{-i(\hat{x}\cdot x')p/\hbar}}{4\pi r} \end{aligned}$$

Now if $k' = \hat{x}|p|/\hbar$, that is k' is set to be in the outgoing x direction to the detector.

$$\rightarrow_{x \gg x'} \quad = \quad -\frac{e^{ikr}}{4\pi r}e^{-ix'\cdot k'}$$

Finally we can write

$$\begin{aligned} \langle x \mid \psi\rangle &= \frac{2m}{\hbar^2}\int d^3x' G(x, x')V(x')\langle x' \mid \psi\rangle + \langle x \mid k\rangle \\ \langle x \mid \psi\rangle &= -\frac{2m}{\hbar^2}\frac{e^{ikr}}{4\pi r}\int d^3x' e^{-ix'\cdot\hat{x}p/\hbar}V(x')\langle x' \mid \psi\rangle + \langle x \mid k\rangle \\ \langle x \mid \psi\rangle &= -\frac{2m}{\hbar^2}\frac{e^{ikr}}{4\pi r}\int d^3x' e^{-ix'\cdot k'}V(x')\langle x' \mid \psi\rangle + \langle x \mid k\rangle \end{aligned}$$

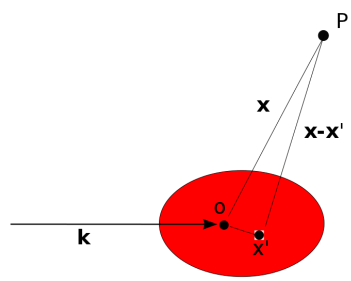

Figure 1.2: Red is the region of the scattering potential. The incident plane wave is headed in the **k** direction. The scattered particle is observed at P. The direction of propagation of the outgoing plane wave is **x**.

$$
\begin{aligned}
&= -\frac{2m}{\hbar^2}\frac{e^{ikr}}{4\pi r}\int d^3x' e^{-ix'\cdot k'} V(x')\langle x' \mid \psi\rangle + \frac{e^{ix\cdot k}}{(2\pi)^{3/2}} \\
&= \frac{1}{(2\pi)^{3/2}}\left(-\frac{e^{ikr}}{r}\frac{2m(2\pi)^3}{4\pi\hbar^2}\int d^3x' \frac{e^{-ix'\cdot k'}}{(2\pi)^{3/2}} V(x')\langle x' \mid \psi\rangle + e^{ix\cdot k}\right)
\end{aligned}
$$

Define the scattering amplitude

$$
f(k', k) = -\frac{4\pi^2 m}{\hbar^2}\int d^3x' \langle k' \mid x'\rangle V(x')\langle x' \mid \psi^+\rangle \tag{1.4}
$$

Then

$$
\langle x \mid \psi\rangle \sim \frac{1}{(2\pi)^{3/2}}\left(\frac{e^{ikr}}{r} f(k', k) + e^{ik\cdot x}\right)
$$

1.4 Differential cross section

We found that the solution to the Schrodinger equation has the form

$$\langle x \mid \psi\rangle \quad \sim \quad \frac{1}{(2\pi)^{3/2}}\left(\frac{e^{ikr}}{r}f(k',k)+e^{ik\cdot x}\right)$$

and that

$$f(\mathbf{k}',\mathbf{k})=\frac{4\pi^2 m}{\hbar^2}\int d^3x'\langle k' \mid x'\rangle V(x')\langle x' \mid \psi\rangle \tag{1.5}$$

Not really much good since we need the solution to do the calculation, but we do learn something about the form. We have been sloppy about normalization. Multiply by $V^{-\frac{1}{2}}$ where V is volume of space. Then if the plane wave represents the incoming flux, we have incident flux $v/V=\hbar k/mV$. The flux scattered radially outward is

$$\frac{v}{V}\frac{|f(k,k')|^2}{r^2}.$$

Let $d\dot{N}$ be the number of particles scattered outward per unit time into the solid angle $d\Omega$.

$$d\dot{N}=\frac{v}{V}\frac{|f(k,k')|^2}{r^2}r^2 d\Omega \tag{1.6}$$

The differential cross section

$$\frac{d\sigma}{d\Omega}=\frac{d\dot{N}}{\text{Inc Flux}}=|f(k,k')|^2 \tag{1.7}$$

1.4.1 Probability current

The scattered particle probability flux is

$$\begin{aligned} j &= \frac{\hbar}{m}\mathrm{Im}(\psi^*\nabla\psi) \\ &\sim \frac{\hbar}{m}\mathrm{Im}\left(\frac{1}{8\pi^3}\frac{f^*}{r}e^{-ikr}\left(ik\frac{e^{ikr}}{r}f-\frac{e^{ikr}}{r^2}f+\frac{e^{ikr}}{r}\nabla f\right)\right)\end{aligned}$$

At large r, all terms fall off faster than $1/r^2$ except the first. Note that ∇f will involve angular derivatives that all have $1/r$ and then derivatives with respect to θ and ϕ. So very far away,

$$\begin{aligned} j &\sim \frac{\hbar}{m}\mathrm{Im}\left(\frac{1}{8\pi^3}\frac{f^*}{r}e^{-ikr}\left(ik\frac{e^{ikr}}{r}f\right)\right) \\ &\sim \frac{\hbar}{m}\frac{k|f^2|}{8\pi^3 r^2}\end{aligned}$$

The flux into the detector with area $r^2 d\Omega$ will be $F_{det}=jr^2 d\Omega$. The total incoming flux is $j_{inc}=k\frac{\hbar}{(2\pi)^3 m}$. Then the rate of scattering into solid angle $d\Omega$ is

$$R=F_{inc}d\sigma=j_{scat}r^2 d\Omega\rightarrow\frac{d\sigma}{d\Omega}=\frac{j^2 r^2}{j_{inc}}=|f|^2$$

That's all well and good. But, probability is conserved. So $\mathbf{j}$ for the entire wave function ψ integrated over the entire sphere must be zero.