February 9, 2015 Lecture VIII

1.2 Differential cross section

We found that the solution to the Schrodinger equation has the form

$$\langle x \mid \psi \rangle \sim \frac{1}{(2\pi)^{3/2}} \left(\frac{e^{ikr}}{r} f(k',k) + e^{ik \cdot x} \right)$$

and that

$$f(\mathbf{k}',\mathbf{k}) = \frac{4\pi^2 m}{\hbar^2} \int d^3 x' \langle k' \mid x' \rangle V(x') \langle x' \mid \psi \rangle$$
(1.1)

Not really much good since we need the solution to do the calculation, but we do learn something about the form. We have been sloppy about normalization. Multiply by $V^{-\frac{1}{2}}$ where V is volume of space. Then if the plane wave represents the incoming flux, we have incident flux $v/V = \hbar k/mV$. The flux scattered radially outward is

$$\frac{v}{V}\frac{|f(k,k')|^2}{r^2}.$$

Let $d\dot{N}$ be the number of particles scattered outward per unit time into the solid angle $d\Omega$.

$$d\dot{N} = \frac{v}{V} \frac{|f(k,k')|^2}{r^2} r^2 d\Omega$$
(1.2)

The differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{dN}{\text{Inc Flux}} = |f(k,k')|^2 \tag{1.3}$$

1.2.1 Probability current

The scattered particle probability flux is

$$j = \frac{\hbar}{m} \operatorname{Im}(\psi^* \nabla \psi)$$

$$\sim \frac{\hbar}{m} \operatorname{Im}\left(\frac{1}{8\pi^3} \frac{f^*}{r} e^{-ikr} \left(ik \frac{e^{ikr}}{r} f - \frac{e^{ikr}}{r^2} f + \frac{e^{ikr}}{r} \nabla f\right)\right)$$

At large r, all terms fall off faster than $1/r^2$ except the first. Note that ∇f will involve angular derivatives that all have 1/r and then derivatives with respect to θ and ϕ . So very far away,

$$j \sim \frac{\hbar}{m} \operatorname{Im} \left(\frac{1}{8\pi^3} \frac{f^*}{r} e^{-ikr} \left(ik \frac{e^{ikr}}{r} f \right) \right)$$
$$\sim \frac{\hbar}{m} \frac{k|f^2|}{8\pi^3 r^2}$$

The flux into the detector with area $r^2 d\Omega$ will be $F_{det} = jr^2 d\Omega$. The total incoming flux is $j_{inc} = k \frac{\hbar}{(2\pi)^3 m}$. Then the rate of scattering into solid angle $d\Omega$ is

$$R = F_{inc}d\sigma = j_{scat}r^2d\Omega \rightarrow \frac{d\sigma}{d\Omega} = \frac{j^2r^2}{j_{inc}} = |f|^2$$

That's all well and good. But, probability is conserved. So **j** for the entire wave function ψ integrated over the entire sphere must be zero.

1.3 Born approximation

We have an integral equation for the scattering amplitude but it is of limited valued since it includes the solution to Schrodinger's equation. The first order Born approximation is pretty simple. We assume that the potential is very weak and that the exact solution is not very different from the free particle state. Then we get

$$f^{1}(k',k) = -\frac{4\pi^{2}m}{\hbar^{2}} \int d^{3}x' \langle k' \mid x' \rangle V(x') \langle x' \mid k \rangle$$
$$= -\frac{4\pi^{2}m}{\hbar^{2}} \int d^{3}x' \frac{e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}'}}{(2\pi)^{3}} V(x')$$
(1.4)

Here, \mathbf{k}, \mathbf{k}' are in the direction of the incoming plane wave and the scattered wave respectively. Define $|\mathbf{q}| = |\mathbf{k} - \mathbf{k}'| = 2k \sin \theta / 2$ where θ is the scattering angle.

Then we can perform the angular integral if we assume that V is spherically symmetric.

$$f^{1}(k',k) = -\frac{4\pi^{2}m}{\hbar^{2}} \int d^{3}x' \frac{e^{i\mathbf{q}\cdot\mathbf{x}'}}{(2\pi)^{3}} V(x')$$
(1.5)

$$= -\frac{4\pi^2 m}{\hbar^2} \int r^2 dr 2\pi (\cos \theta') \frac{e^{iqr \cos \theta'}}{(2\pi)^3} V(r)$$
(1.6)

$$= -\frac{4\pi^2 m}{\hbar^2} \int r^2 dr 2\pi (\cos \theta') \frac{e^{iqr \cos \theta'}}{(2\pi)^3} V(r)$$
(1.7)

$$= -\frac{4\pi^2 m}{iq\hbar^2} \int \frac{r^2}{r} dr 2\pi \frac{e^{iqr} - e^{-iqr}}{(2\pi)^3} V(r)$$
(1.8)

$$= -\frac{m}{iq\hbar^2} \int r dr (e^{iqr} - e^{-iqr}) V(r)$$
(1.9)

$$= -\frac{2m}{q\hbar^2} \int r dr \sin(qr) V(r) \tag{1.10}$$

Yukawa potential

Consider the Yukawa potential

$$V(r) = \frac{V_0 e^{-\mu r}}{\mu r}$$

which reduces to the Coulomb potential with $\mu \to 0$ with V_0/μ fixed. Substitution and integration gives

$$f^{1}(\theta) = -\left(\frac{2mV_{0}}{\mu\hbar^{2}}\right)\frac{1}{q^{2}+\mu^{2}}$$
$$= -\left(\frac{2mV_{0}}{\mu\hbar^{2}}\right)\frac{1}{4k^{2}\sin^{2}(\theta/2)+\mu^{2}}$$

Note that for the first order Born approximation, the scattering cross section is always independent of the sign of V(r), and the scattering amplitude is always real

1.3.1 Higher Order Born Approximation and transition operator T

We would like to have an operator that effects a transition from a plane wave initial state to a plane wave final state. Let's revisit the Schrödinger equation for the plane wave.

$$H_{0} | \phi \rangle = E | \phi \rangle$$

$$(H - V) | \phi \rangle = E | \phi \rangle$$

$$(E - H) | \phi \rangle = -V | \phi \rangle$$
(1.11)

$$|\phi\rangle = -\frac{1}{E - H \pm i\epsilon} V |\phi\rangle + |\psi^{\pm}\rangle$$
(1.12)

where $(E - H) | \psi^{\pm} \rangle = 0$. Solve Equation ?? for

$$|\psi^{\pm}\rangle = \left(1 + \frac{1}{E - H \pm i\epsilon}V\right)|\phi\rangle$$

and

$$V | \psi^{\pm} \rangle = \left(V + V \frac{1}{E - H \pm i\epsilon} V \right) | \phi \rangle$$

Then the transition operator

$$T = \left(V + V \frac{1}{E - H \pm i\epsilon} V\right) | \phi\rangle$$

$$T | \phi(k) \rangle = V | \psi\rangle$$
 (1.13)

where $|\phi(k)\rangle$ is a plane wave with momentum k, and $|\psi\rangle$ is a solution to Schrödinger's equation. The differential cross section

$$f(\mathbf{k}',\mathbf{k}) = -\frac{4\pi^2 m}{\hbar^2} \left\langle \phi(\mathbf{k}') \mid T \mid \phi(\mathbf{k}) \right.$$

1.3.2 Born approximation again

Multiplying (Lippmann-Schwinger) Equation by V gives

$$T \mid \phi \rangle = V \frac{1}{E - H_0} T \mid \phi \rangle + V \mid \phi \rangle$$

Assuming that latter is true for a complete set of base states, it must be a legitimate operator equation.

$$T = V \frac{1}{E - H_0 + i\epsilon} T + V \tag{1.14}$$

On iteration we get something like

$$T = V + V \frac{1}{E - H_0 + i\epsilon} \left(V + V \frac{1}{E - H_0 + i\epsilon} T \right)$$

$$\rightarrow V + V \frac{1}{E - H_0 + i\epsilon} \left(V + V \frac{1}{E - H_0 + i\epsilon} \left(V + V \frac{1}{E - H_0 + i\epsilon} T \right) \right)$$

$$= V + V \frac{1}{E - H_0 + i\epsilon} V + V \frac{1}{E - H_0 + i\epsilon} V \frac{1}{E - H_0 + i\epsilon} V + \dots$$

and so on. The scattering amplitude

$$f(\mathbf{k}', \mathbf{k}) = -\frac{4\pi^2 m}{\hbar^2} \langle \mathbf{k}' \mid V \mid \psi \rangle$$
(1.15)

becomes

$$f(\mathbf{k}', \mathbf{k}) = -\frac{4\pi^2 m}{\hbar^2} \langle \mathbf{k}' \mid T \mid \mathbf{k} \rangle$$
(1.16)

The momentum eigneket $|\mathbf{k}\rangle$ is scattered to definite momentum plane wave state $|\mathbf{k}'\rangle$.

Then

$$f^{1}(\mathbf{k}',\mathbf{k}) = -\frac{4\pi^{2}m}{(2\pi)^{3}\hbar^{2}} \int d^{3}x' e^{-i\mathbf{k}'\cdot\mathbf{x}'} V(\mathbf{x}') e^{i\mathbf{x}'\cdot\mathbf{k}}$$
(1.17)

Next order

$$f^{2}(\mathbf{k}',\mathbf{k}) = -\frac{4\pi^{2}m}{(2\pi)^{3}\hbar^{2}} \int d^{3}x' \int d^{3}\mathbf{x}'' e^{-i\mathbf{k}'\cdot\mathbf{x}'} V(\mathbf{x}') \frac{2m}{\hbar^{2}} G(\mathbf{x}',\mathbf{x}'') V(\mathbf{x}'') e^{i\mathbf{x}''\cdot\mathbf{k}}$$
(1.18)

1.4 Currents and optical theorem

$$\psi = \psi^0 + \psi_s \tag{1.19}$$

 ψ^0 represents the incoming free particle and is a solution to

$$H^0\psi^0 = i\hbar\frac{\partial}{\partial t}\psi_0$$

where $H_0 = \frac{p^2}{2m}$. Then $H\psi = i\hbar \frac{\partial}{\partial t}\psi$ where $H = H^0 + V$ and The flux of scattered particles into area element da is

$$\mathbf{j}_{\mathbf{s}} \cdot \hat{\mathbf{n}} da = \frac{\hbar}{m} \operatorname{Im}(\psi_{s}^{*} \nabla \psi_{s}) \cdot \hat{n} da$$
$$= \frac{k}{8\pi^{3}} \frac{\hbar}{m} \frac{|f|^{2}}{r^{2}} r^{2} d\Omega$$

The flux of incoming particles is

$$\mathbf{j_{inc}} = \frac{\hbar}{m} \mathrm{Im}(\psi_0^* \nabla \psi_0) = \frac{k}{8\pi^3} \frac{\hbar}{m}$$
(1.20)

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\mathbf{j}_{\mathbf{scat}} \cdot \hat{\mathbf{n}} da}{\mathbf{j}_{\mathbf{inc}}} = |f|^2 \tag{1.21}$$

Then along with the divergence theorem,

$$\sigma_t = \frac{8\pi^3 m}{\hbar k} \int \mathbf{j}_{\mathbf{s}} \cdot \hat{\mathbf{n}} da = \frac{8\pi^3 m}{\hbar k} \int \nabla \cdot \mathbf{j}_{\mathbf{s}} dv$$

By the continuity equation

$$\int \nabla \cdot \mathbf{j}_{\mathbf{s}} dv = \int \frac{\partial}{\partial t} |\psi_s|^2 dv$$

Altogether we find that

$$\sigma_t = \frac{m}{\hbar k} \int \frac{\partial}{\partial t} |\psi_s|^2 dv$$

Substituting Equation 1.1 we have

$$\begin{split} \sigma_t &= \frac{8\pi^3 m}{\hbar k} \frac{\partial}{\partial t} \int dv \left(|\psi_0|^2 + |\psi|^2 - 2\Re\psi_0^*\psi \right) \\ &= \frac{8\pi^3 m}{\hbar k} \int dv \left(\frac{\partial\psi_0^*}{\partial t} \psi_0 + \psi_0^* \frac{\partial\psi_0}{\partial t} + \frac{\partial\psi^*}{\partial t} \psi + \psi^* \frac{\partial\psi}{\partial t} - 2\Re(\frac{\partial\psi_0^*}{\partial t} \psi + \psi_0^* \frac{\partial\psi}{\partial t}) \right) \\ &= -2\frac{8\pi^3 m}{\hbar k} \Re \int \frac{i}{\hbar} dv (-H_0 \psi_0^* \psi + \psi_0^* (H_0 + V)\psi) \\ &= -2\frac{8\pi^3 m}{\hbar k} \Re \int \frac{i}{\hbar} dv (\psi_0^* V\psi) \end{split}$$

$$= \operatorname{Im} \frac{8\pi^3 m}{\hbar k} \frac{2}{\hbar} \langle \mathbf{k} \mid V \mid \psi \rangle$$
$$= \operatorname{Im} \frac{8\pi^3 m}{\hbar k} \frac{2}{\hbar} \frac{\hbar^2}{4\pi^2 m} f(0)$$
$$= \operatorname{Im} \frac{4\pi}{k} f(0)$$

The total cross section is proportional to the imaginary part of the forward scattering amplitude. The flux of scattered particles is balanced by the imaginary part of the forward amplitude, the shadow.

There is another way: The total wave function

$$\psi_{inc} + \psi_{scat} = e^{ikz} + f(\theta) \frac{e^{ikr}}{r}$$
(1.22)

The flux density is

$$\mathbf{j} = \frac{\hbar}{m} \mathrm{Im}(\psi^* \nabla \psi) \tag{1.23}$$

The total flux density in the radial direction is

$$j_r = \frac{\hbar}{m} \operatorname{Im} \left((e^{-ikz} + f^* \frac{e^{-ikr}}{r}) (ik\cos\theta e^{ikr\cos\theta} + (ikf - \frac{1}{r^2}) \frac{e^{ikr}}{r}) \hat{\mathbf{r}} \right)$$

Since we are interested in $r \to \infty$, only the first and two terms in the second brackets will remain. Then

$$j_r = \frac{\hbar}{m} \operatorname{Im} \left((e^{-ikz} + f^* \frac{e^{-ikr}}{r}) (ik\cos\theta e^{ikr\cos\theta} + ikf) \frac{e^{ikr}}{r} \right) \hat{\mathbf{r}}$$

The interference term is

$$j_r^{int} = \frac{\hbar}{m} \operatorname{Im} \frac{ik}{r} \left(e^{-ikr(\cos\theta - 1)} f + f^* e^{-ikr(1 - \cos\theta} \cos\theta \right) \hat{\mathbf{r}} \right]$$

Next integrate j_r^{int} over a tiny cone in the forward direction to show

$$\int_{\text{forward cone}} j_r^{int} r^2 d\Omega = -\left(\frac{\hbar k}{m}\right) \frac{4\pi}{k} \text{Im} f(0)$$
(1.24)

$$\begin{split} \int_{\text{forward cone}} j_r^{int} r^2 d\Omega &= 2\pi \frac{\hbar rk}{m} \int_0^\beta \text{Im}i(e^{-ikr(\cos\theta - 1)}f + \cos\theta f^* e^{-ikr(1 - \cos\theta)})d(\cos\theta) \\ &\sim 2\pi \frac{\hbar rk}{m} \int_0^\beta \text{Im}i(e^{ikr} e^{-ikr\cos\theta}f + f^* e^{-ikr} e^{ikr\cos\theta})d(\cos\theta) \\ &= 2\pi \frac{\hbar rk}{m} \text{Im}i(\frac{e^{ikr}}{-ikr} e^{-ikr\cos\theta}f + f^* \frac{e^{-ikr}}{ikr} e^{ikr\cos\theta}) \mid_0^\beta \end{split}$$

$$= 2\pi \frac{\hbar r k}{m} \operatorname{Im} i(\frac{e^{ikr}}{-ikr} (e^{-ikr\cos\beta} - 1)f + f^* \frac{e^{-ikr}}{ikr} (e^{ikr\cos\beta} - 1)) \\ = 2\pi \frac{\hbar r k}{m} \operatorname{Im} (\frac{1}{-kr} (e^{-ikr(\cos\beta - 1)} - e^{ikr})f + f^* \frac{1}{kr} (e^{ikr(\cos\beta - 1)} - e^{-ikr}))$$

As long as $\theta \neq 0$, the average of j_r^{int} over any small solid angle is zero because $r \to \infty$. (Assume $f(\theta)$ is a smooth function.)

In the limit $\beta \to 0$, and as $r \to \infty$, we use the average value for $e^{\pm irk}$, namely zero, we get

$$\int_{\text{forward cone}} j_r^{int} r^2 d\Omega = 2\pi \frac{\hbar k}{m} r \text{Im}(\frac{1}{-kr}(1)f + f^* \frac{1}{kr}(1))$$
$$= 2\pi \frac{\hbar k}{m} \frac{1}{k} \text{Im}(f^* - f)$$
$$= -4\pi \frac{\hbar k}{m} \frac{1}{k} \text{Im}f(0)$$
$$= -4\pi \frac{\hbar k}{m} \frac{1}{k} \text{Im}f(0)$$

In evaluating the upper limit in the θ integration, assume that the limit of a function that oscillates as its argument approaches infinity is equal to its average value.

The total probability current in the region behind the target produces a depletion of particles. It must be that the product of the incident flux and the total cross section is equivalent to hwat is depleted in the forward direction. Therefore

$$\int_{forward} j_r^{int} r^2 d\Omega = -4\pi \frac{\hbar k}{m} \frac{1}{k} \text{Im} f(0) = \frac{\hbar k}{m} \sigma_t$$
$$\rightarrow \sigma_t = \frac{4\pi}{k} \text{Im} f(0)$$

1.4.1 Optical Theorem

We begin with the basic Lippmann/Schwinger equation

$$|\psi^{\pm}\rangle = |\phi\rangle + \frac{1}{E - H_0 \pm i\epsilon} V|\psi^{\pm}\rangle$$
(1.25)

The scattering amplitude is

$$f(\mathbf{k}, \mathbf{k}') = -\frac{4\pi^2 m}{\hbar^2} \left\langle \phi(\mathbf{k}) \mid T \mid \phi(\mathbf{k}') \right\rangle$$
(1.26)

where $T \mid \phi \rangle = V \mid \psi^{\pm} \rangle$ by definition. Then

$$f(\mathbf{k}, \mathbf{k}') = \left(\langle \psi^{\pm} | - \langle \psi^{\pm} | V \frac{1}{E - H_0 \pm i\epsilon} \right) V | \psi^{\pm} \rangle$$
$$= \langle \psi^{\pm} | V | \psi^{\pm} \rangle - \langle \psi^{\pm} | V \frac{1}{E - H_0 \pm i\epsilon} V | \psi^{\pm} \rangle$$

The imaginary part of the forward scattering amplitude is

$$\begin{split} \operatorname{Im} f(\mathbf{k}, \mathbf{k}) &= -\frac{8\pi^{3}}{4\pi} \frac{2m}{\hbar^{2}} \langle \mathbf{k} \mid T \mid \mathbf{k} \rangle \\ \operatorname{Im} f(\mathbf{k}, \mathbf{k}) &= -\operatorname{Im} \frac{8\pi^{3}}{4\pi} \frac{2m}{\hbar^{2}} \int \int \langle \phi(\mathbf{k}) \mid T \mid \phi_{k'} \rangle \langle \phi_{k'} \mid \frac{1}{E - H_{0} \pm i\epsilon} \mid \phi_{k''} \rangle \langle \phi_{k''} \mid T \mid \phi(\mathbf{k}) \rangle d^{3}k' d^{3}k'' \\ \operatorname{Im} f(\mathbf{k}, \mathbf{k}) &= -\operatorname{Im} \frac{8\pi^{3}}{4\pi} \frac{2m}{\hbar^{2}} \frac{2m}{\hbar^{2}} \int d^{3}k' \langle \phi(\mathbf{k}) \mid T \mid \phi_{k'} \rangle \frac{1}{k''^{2} - k'^{2} \pm i\epsilon} \langle \phi_{k'} \mid T \mid \phi(\mathbf{k}) \rangle \\ &= -\operatorname{Im} \frac{8\pi^{3}}{4\pi} \frac{2m}{\hbar^{2}} \frac{2m}{\hbar^{2}} \int dk' k'^{2} d\Omega \langle \phi(\mathbf{k}) \mid T \mid \phi_{k'} \rangle \frac{1}{k''^{2} - k'^{2} \pm i\epsilon} \langle \phi_{k'} \mid T \mid \phi(\mathbf{k}) \rangle \\ &= -\operatorname{Im} \frac{8\pi^{3}}{4\pi} \frac{2m}{\hbar^{2}} \frac{2m}{\hbar^{2}} \int dk' k'^{2} d\Omega \langle \phi(\mathbf{k}) \mid T \mid \phi_{k'} \rangle \frac{1}{(k''\sqrt{1 \pm \frac{i\epsilon}{k''^{2}}} + k')(k''\sqrt{1 \pm \frac{i\epsilon}{k''^{2}}} - k')} \langle \phi_{k'} \mid T \mid \phi(\mathbf{k}) \rangle \\ &\sim -\operatorname{Im} \frac{8\pi^{3}}{4\pi} \frac{2m}{\hbar^{2}} \frac{2m}{\hbar^{2}} \int dk' k'^{2} d\Omega \langle \phi(\mathbf{k}) \mid T \mid \phi_{k'} \rangle \frac{1}{(k'' \pm i\epsilon + k')(k'' \pm i\epsilon - k'))} \langle \phi_{k'} \mid T \mid \phi(\mathbf{k}) \rangle \\ &\sim -\operatorname{Im} i\pi \frac{1}{2\pi^{2}} \frac{8\pi^{3}}{4\pi} \frac{8\pi^{3}}{\hbar^{2}} \frac{2m}{\hbar^{2}} \int k''^{2} d\Omega \langle \phi(\mathbf{k}) \mid T \mid \phi_{k'} \rangle \frac{1}{2k''} \langle \phi_{k''} \mid T \mid \phi(\mathbf{k}) \rangle \\ &\sim -\operatorname{Im} i\pi \frac{1}{2\pi^{2}} \frac{4\pi^{3}}{4\pi} \frac{8\pi^{3}}{\hbar^{2}} \frac{2m}{\hbar^{2}} \int k''^{2} d\Omega \langle \phi(\mathbf{k}) \mid T \mid \phi_{k''} \rangle \frac{1}{2k''} \langle \phi_{k''} \mid T \mid \phi(\mathbf{k}) \rangle \\ &\sim -\operatorname{Im} \frac{ik''}{4\pi} \int d\Omega f^{*}(\mathbf{k}, \mathbf{k}'') f(\mathbf{k}'', \mathbf{k}) \\ &\sim -\frac{k''}{4\pi} \sigma_{i} \end{aligned}$$

Shadowing

We write the solution to Schrodinger's equation as ψ^+ as the sum of an incoming plane wave that extends over all space, and an outgoing spherical wave with angular distribution represented as $f(\theta)$. Consider scattering from a hard sphere. The forward direction along the axis of the incoming wave is shadowed. In that region the probability density $|\psi^+|^2 = 0$. There must be destructive interference between the incoming plane wave and the scattering amplitude in the forward direction. So the scattering amplitude in the forward direction cannot be zero.

More generally we write

$$\sigma_{tot} = \frac{4\pi}{k} \mathrm{Im} f_{elastic}(0) \tag{1.27}$$

Only states scattered elastically in the forward direction will have the same energy as the incident state, which is required if there is to be destructive interference. Also the depletion of the forward flux must account for all scattered states elastic or inelastic.