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Lecture IX

1.2 Currents and optical theorem

$$\psi = \psi^0 + \psi_s \tag{1.1}$$

ψ^0 represents the incoming free particle and is a solution to

$$H^0\psi^0 = i\hbar\frac{\partial}{\partial t}\psi_0$$

where $H_0 = \frac{p^2}{2m}$. Then $H\psi = i\hbar\frac{\partial}{\partial t}\psi$ where $H = H^0 + V$ and

The flux of scattered particles into area element da is

$$\begin{aligned}\mathbf{j_s}\cdot\hat{\mathbf{n}}da &= \frac{\hbar}{m}\mathrm{Im}(\psi_s^*\nabla\psi_s)\cdot\hat{n}da \\ &= \frac{k}{8\pi^3}\frac{\hbar}{m}\frac{|f|^2}{r^2}r^2d\Omega\end{aligned}$$

The flux of incoming particles is

$$\mathbf{j_{inc}} = \frac{\hbar}{m}\mathrm{Im}(\psi_0^*\nabla\psi_0) = \frac{k}{8\pi^3}\frac{\hbar}{m} \tag{1.2}$$

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\mathbf{j_{scat}}\cdot\hat{\mathbf{n}}da}{\mathbf{j_{inc}}} = |f|^2 \tag{1.3}$$

Then along with the divergence theorem,

$$\sigma_t = \frac{8\pi^3 m}{\hbar k}\int\mathbf{j_s}\cdot\hat{\mathbf{n}}da \quad = \quad \frac{8\pi^3 m}{\hbar k}\int\nabla\cdot\mathbf{j_s}dv$$

By the continuity equation

$$\int\nabla\cdot\mathbf{j_s}dv \quad = \quad \int\frac{\partial}{\partial t}|\psi_s|^2dv$$

Altogether we find that

$$\sigma_t = \frac{m}{\hbar k}\int\frac{\partial}{\partial t}|\psi_s|^2dv$$

Substituting Equation 1.1 we have

$$\begin{aligned}\sigma_t &= \frac{8\pi^3 m}{\hbar k}\frac{\partial}{\partial t}\int dv\left(|\psi_0|^2 + |\psi|^2 - 2\Re\psi_0^*\psi\right) \\ &= \frac{8\pi^3 m}{\hbar k}\int dv\left(\frac{\partial\psi_0^*}{\partial t}\psi_0 + \psi_0^*\frac{\partial\psi_0}{\partial t} + \frac{\partial\psi^*}{\partial t}\psi + \psi^*\frac{\partial\psi}{\partial t} - 2\Re(\frac{\partial\psi_0^*}{\partial t}\psi + \psi_0^*\frac{\partial\psi}{\partial t})\right)\end{aligned}$$

$$
\begin{aligned}
&= -2\frac{8\pi^3 m}{\hbar k}\Re\int \frac{i}{\hbar}dv(-H_0\psi_0^*\psi + \psi_0^*(H_0+V)\psi) \\
&= -2\frac{8\pi^3 m}{\hbar k}\Re\int \frac{i}{\hbar}dv(\psi_0^* V\psi) \\
&= \mathrm{Im}\frac{8\pi^3 m}{\hbar k}\frac{2}{\hbar}\langle \mathbf{k} \mid V \mid \psi\rangle \\
&= \mathrm{Im}\frac{8\pi^3 m}{\hbar k}\frac{2}{\hbar}\frac{\hbar^2}{4\pi^2 m}f(0) \\
&= \mathrm{Im}\frac{4\pi}{k}f(0)
\end{aligned}
$$

The total cross section is proportional to the imaginary part of the forward scattering amplitude. The flux of scattered particles is balanced by the imaginary part of the forward amplitude, the shadow.

There is another way: The total wave function

$$\psi_{inc} + \psi_{scat} = e^{ikz} + f(\theta)\frac{e^{ikr}}{r} \tag{1.4}$$

The flux density is

$$\mathbf{j} = \frac{\hbar}{m}\mathrm{Im}(\psi^*\nabla\psi) \tag{1.5}$$

The total flux density in the radial direction is

$$j_r = \frac{\hbar}{m}\mathrm{Im}\left((e^{-ikz} + f^*\frac{e^{-ikr}}{r})(ik\cos\theta e^{ikr\cos\theta} + (ikf - \frac{1}{r^2})\frac{e^{ikr}}{r})\hat{\mathbf{r}}\right)$$

Since we are interested in $r \to \infty$, only the first and two terms in the second brackets will remain. Then

$$j_r = \frac{\hbar}{m}\mathrm{Im}\left((e^{-ikz} + f^*\frac{e^{-ikr}}{r})(ik\cos\theta e^{ikr\cos\theta} + ikf)\frac{e^{ikr}}{r}\right)\hat{\mathbf{r}}$$

The interference term is

$$j_r^{int} = \frac{\hbar}{m}\mathrm{Im}\frac{ik}{r}\left(e^{-ikr(\cos\theta-1)}f + f^*e^{-ikr(1-\cos\theta}\cos\theta\right)\hat{\mathbf{r}}]$$

Next integrate j_r^{int} over a tiny cone in the forward direction to show

$$\int_{\text{forward cone}} j_r^{int} r^2 d\Omega = -\left(\frac{\hbar k}{m}\right)\frac{4\pi}{k}\mathrm{Im}f(0) \tag{1.6}$$

$$\int_{\text{forward cone}} j_r^{int} r^2 d\Omega = 2\pi\frac{\hbar r k}{m}\int_0^\beta \mathrm{Im} i(e^{-ikr(\cos\theta-1)}f + \cos\theta f^* e^{-ikr(1-\cos\theta)})d(\cos\theta)$$

$$
\begin{aligned}
&\sim \quad 2\pi\frac{\hbar rk}{m}\int_0^\beta \mathrm{Im}i(e^{ikr}e^{-ikr\cos\theta}f + f^*e^{-ikr}e^{ikr\cos\theta})d(\cos\theta) \\
&= \quad 2\pi\frac{\hbar rk}{m}\mathrm{Im}i(\frac{e^{ikr}}{-ikr}e^{-ikr\cos\theta}f + f^*\frac{e^{-ikr}}{ikr}e^{ikr\cos\theta})\mid_0^\beta \\
&= \quad 2\pi\frac{\hbar rk}{m}\mathrm{Im}i(\frac{e^{ikr}}{-ikr}(e^{-ikr\cos\beta}-1)f + f^*\frac{e^{-ikr}}{ikr}(e^{ikr\cos\beta}-1)) \\
&= \quad 2\pi\frac{\hbar rk}{m}\mathrm{Im}(\frac{1}{-kr}(e^{-ikr(\cos\beta-1)}-e^{ikr})f + f^*\frac{1}{kr}(e^{ikr(\cos\beta-1)}-e^{-ikr}))
\end{aligned}
$$

As long as $\theta \neq 0$, the average of j_r^{int} over any small solid angle is zero because $r \to \infty$. (Assume $f(\theta)$ is a smooth function.)

In the limit $\beta \to 0$, and as $r \to \infty$, we use the average value for $e^{\pm irk}$, namely zero, we get

$$
\begin{aligned}
\int_{\text{forward cone}} j_r^{int} r^2 d\Omega &= 2\pi\frac{\hbar k}{m}r\mathrm{Im}(\frac{1}{-kr}(1)f + f^*\frac{1}{kr}(1)) \\
&= 2\pi\frac{\hbar k}{m}\frac{1}{k}\mathrm{Im}(f^* - f) \\
&= -4\pi\frac{\hbar k}{m}\frac{1}{k}\mathrm{Im}f(0) \\
&= -4\pi\frac{\hbar k}{m}\frac{1}{k}\mathrm{Im}f(0)]
\end{aligned}
$$

In evaluating the upper limit in the θ integration, assume that the limit of a function that oscillates as its argument approaches infinity is equal to its average value.

The total probability current in the region behind the target produces a depletion of particles. It must be that the product of the incident flux and the total cross section is equivalent to hwat is depleted in the forward direction. Therefore

$$
\begin{aligned}
\int_{forward} j_r^{int} r^2 d\Omega &= -4\pi\frac{\hbar k}{m}\frac{1}{k}\mathrm{Im}f(0) = \frac{\hbar k}{m}\sigma_t \\
\to \sigma_t = \frac{4\pi}{k}\mathrm{Im}f(0)
\end{aligned}
$$

1.2.1 Optical Theorem

We begin with the basic Lippmann/Schwinger equation

$$
\mid \psi^\pm\rangle = \mid \phi\rangle + \frac{1}{E - H_0 \pm i\epsilon}V\mid \psi^\pm\rangle \tag{1.7}
$$

The scattering amplitude is

$$
f(\mathbf{k}, \mathbf{k}') = -\frac{4\pi^2 m}{\hbar^2}\langle\phi(\mathbf{k}) \mid T \mid \phi(\mathbf{k}')\rangle \tag{1.8}
$$

where $T|\,\phi\rangle = V|\,\psi^{\pm}\rangle$ by definition. Then

$$\begin{aligned} f(\mathbf{k},\mathbf{k}') &= \left(\langle\psi^{\pm}\,| - \langle\psi^{\pm}\,|V\frac{1}{E-H_0\pm i\epsilon}\right)V|\,\psi^{\pm}\rangle \\ &= \langle\psi^{\pm}\,|V|\,\psi^{\pm}\rangle - \langle\psi^{\pm}\,|V\frac{1}{E-H_0\pm i\epsilon}V|\,\psi^{\pm}\rangle \end{aligned}$$

The imaginary part of the forward scattering amplitude is

$$\begin{aligned} \mathrm{Im} f(\mathbf{k},\mathbf{k}) &= -\frac{8\pi^3}{4\pi}\frac{2m}{\hbar^2}\langle\mathbf{k}\mid T\mid\mathbf{k}\rangle \\ \mathrm{Im} f(\mathbf{k},\mathbf{k}) &= -\mathrm{Im}\frac{8\pi^3}{4\pi}\frac{2m}{\hbar^2}\int\int\langle\phi(\mathbf{k})\,|T|\,\phi_{k'}\rangle\langle\phi_{k'}\,|\frac{1}{E-H_0\pm i\epsilon}|\,\phi_{k''}\rangle\langle\phi_{k''}\,|T|\,\phi(\mathbf{k})\rangle d^3k'd^3k'' \\ \mathrm{Im} f(\mathbf{k},\mathbf{k}) &= -\mathrm{Im}\frac{8\pi^3}{4\pi}\frac{2m}{\hbar^2}\frac{2m}{\hbar^2}\int d^3k'\langle\phi(\mathbf{k})\,|T|\,\phi_{k'}\rangle\frac{1}{k''^2-k'^2\pm i\epsilon}\langle\phi_{k'}\,|T|\,\phi(\mathbf{k})\rangle \\ &= -\mathrm{Im}\frac{8\pi^3}{4\pi}\frac{2m}{\hbar^2}\frac{2m}{\hbar^2}\int dk'k'^2d\Omega\langle\phi(\mathbf{k})\,|T|\,\phi_{k'}\rangle\frac{1}{k''^2-k'^2\pm i\epsilon}\langle\phi_{k'}\,|T|\,\phi(\mathbf{k})\rangle \\ &= -\mathrm{Im}\frac{8\pi^3}{4\pi}\frac{2m}{\hbar^2}\frac{2m}{\hbar^2}\int dk'k'^2d\Omega\langle\phi(\mathbf{k})\,|T|\,\phi_{k'}\rangle\frac{1}{(k''\sqrt{1\pm\frac{i\epsilon}{k''^2}}+k')(k''\sqrt{1\pm\frac{i\epsilon}{k''^2}}-k')}\langle\phi_{k'}\,|T|\,\phi(\mathbf{k})\rangle \\ &\sim -\mathrm{Im}\frac{8\pi^3}{4\pi}\frac{2m}{\hbar^2}\frac{2m}{\hbar^2}\int dk'k'^2d\Omega\langle\phi(\mathbf{k})\,|T|\,\phi_{k'}\rangle\frac{1}{(k''\pm i\epsilon+k')(k''\pm i\epsilon-k'))}\langle\phi_{k'}\,|T|\,\phi(\mathbf{k})\rangle \\ &\sim -\mathrm{Im}\, i\pi\frac{1}{2\pi^2}\frac{8\pi^3}{4\pi}\frac{8\pi^3}{4\pi}\frac{2m}{\hbar^2}\frac{2m}{\hbar^2}\int k''^2d\Omega\langle\phi(\mathbf{k})\,|T|\,\phi_{k''}\rangle\frac{1}{2k''}\langle\phi_{k''}\,|T|\,\phi(\mathbf{k})\rangle \\ &\sim -\mathrm{Im}\frac{ik''}{4\pi}\int d\Omega f^*(\mathbf{k},\mathbf{k}'')f(\mathbf{k}'',\mathbf{k}) \\ &\sim -\frac{k''}{4\pi}\sigma_t \end{aligned}$$

Shadowing

We write the solution to Schrodinger's equation as ψ^+ as the sum of an incoming plane wave that extends over all space, and an outgoing spherical wave with angular distribution represented as $f(\theta)$. Consider scattering from a hard sphere. The forward direction along the axis of the incoming wave is shadowed. In that region the probability density $|\psi^+|^2 = 0$. There must be destructive interference between the incoming plane wave and the scattering amplitude in the forward direction. So the scattering amplitude in the forward direction cannot be zero.

More generally we write

$$\sigma_{tot} = \frac{4\pi}{k}\mathrm{Im} f_{elastic}(0) \tag{1.9}$$

Only states scattered elastically in the forward direction will have the same energy as the incident state, which is required if there is to be destructive interference. Also the depletion of the forward flux must account for all scattered states elastic or inelastic.

1.3 Partial Wave Analysis

We have described scattering in terms of an incoming plane wave, a momentum eigenket, and and outgoing spherical wave, also with definite momentum. We now consider the basis of free particle states with definite energy and angular momentum (rather than linear momentum) that look like $\mid E, l, m\rangle$. These are eigenkets of of H_0, L^2, and L_z. We would like to expand our plane wave in terms of these spherical waves like so

$$\mid \mathbf{k}\rangle = \sum_{l,m} \mid E,l,m\rangle\langle E,l,m \mid \mathbf{k}\rangle \tag{1.10}$$

Then we can write the scattering amplitude

$$\begin{aligned} f(\mathbf{k}',\mathbf{k}) &= -\frac{1}{4\pi}\frac{2m}{\hbar^2}8\pi^3 \langle \mathbf{k}' \mid T \mid \mathbf{k}\rangle \\ &= -\frac{1}{4\pi}\frac{2m}{\hbar^2}8\pi^3 \int dE' \sum_{l',m'} \int dE \sum_{l,m} \langle \mathbf{k}' \mid E,l',m'\rangle \langle E,l',m' \mid T \mid E,l,m\rangle \langle E,l,m \mid \mathbf{k}\rangle \end{aligned}$$

If the scattering potential is spherically symmetric, T is a scalar operator, and by WE, $l = l', m = m'$, and $\langle E,l,m \mid T \mid E,l,m\rangle$ is independent of m. Then

$$\begin{aligned} f(\mathbf{k}',\mathbf{k}) &= -\frac{1}{4\pi}\frac{2m}{\hbar^2}8\pi^3 \int\int dEdE' \sum_{l,m} \langle \mathbf{k}' \mid E',l,m\rangle \langle E',l,m \mid T \mid E,l,m\rangle \langle E,l,m \mid \mathbf{k}\rangle \\ &= -\frac{1}{4\pi}\frac{2m}{\hbar^2}8\pi^3 \int\int dEdE' \sum_{l,m} \langle \mathbf{k}' \mid E',l,m\rangle T_l \langle E,l,m \mid \mathbf{k}\rangle \qquad (1.11) \\ & \qquad (1.12) \end{aligned}$$

The ”spherical” scattering amplitude conserves angular momentum.

Now let’s figure out $\langle \mathbf{k} \mid E,l,m\rangle$. Consider the state $\mid k\hat{\mathbf{z}}\rangle$.

$$\begin{aligned} \langle k\hat{\mathbf{z}} \mid L_z \mid E,l,m\rangle &= 0 \quad (m \neq 0) \\ \rightarrow \langle k\hat{\mathbf{z}} \mid E,l,m\rangle &= 0 \quad (m \neq 0) \end{aligned}$$

Also $\langle k\hat{\mathbf{z}} \mid E,l,m=0\rangle$ is independent of θ,ϕ,so $\langle k,\hat{\mathbf{z}} \mid E,l,m=0\rangle = \sqrt{\frac{2l+1}{4\pi}}g_l(k)$. We can transform the z-direction momentum ket into an arbitrary direction by a rotation.

$$\mid \mathbf{k}\rangle = \mathcal{D}(\alpha=\phi,\beta=\theta,0)\mid k\hat{\mathbf{z}}\rangle \tag{1.13}$$

Then

$$\begin{aligned} \langle \mathbf{k} \mid E,l,m\rangle &= \langle k\hat{\mathbf{z}} \mid \mathcal{D} \mid E,l,m\rangle \\ &= \sum_{l'} \langle k\hat{\mathbf{z}} \mid E,l',m'=0\rangle \langle E,l',m'=0 \mid \mathcal{D} \mid E,l,m\rangle \\ &= \sum_{l'} \sqrt{\frac{2l+1}{4\pi}} g_l(k)\mathcal{D}^l_{0,m} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{l'} \sqrt{\frac{2l+1}{4\pi}} g_l(k) \sqrt{\frac{4\pi}{2l+1}} Y_l^m(\theta,\phi) \\
&= \sum_l g_l(k) Y_l^m(\theta,\phi)
\end{aligned}
$$

One more thing.

$$
\begin{aligned}
\langle \mathbf{k} \mid H_0 - E \mid E, l, m\rangle &= \langle \mathbf{k} \mid E, l, m\rangle (\frac{\hbar^2 k^2}{2m} - E) = 0 \\
\rightarrow \langle \mathbf{k} \mid E, l, m\rangle &\propto \delta(\frac{\hbar^2 k^2}{2m} - E) \\
\rightarrow g_l(k) &= N_l \delta(\frac{\hbar^2 k^2}{2m} - E)
\end{aligned}
$$

To determine N_l let's try to normalize.

$$
\begin{aligned}
\langle E', l', m' \mid E, l, m\rangle &= \int d^3k \langle E', l', m' \mid \mathbf{k}\rangle\langle \mathbf{k} \mid E, l, m\rangle \\
&= \int k''^2 dk'' d\Omega N_{l'}^* N_l Y_{l'}^{m'*} Y_l^m \delta(\frac{\hbar^2 k''^2}{2m} - E')\delta(\frac{\hbar^2 k''^2}{2m} - E) \\
&= \int k''^2 dk'' |N_l|^2 \delta(\frac{\hbar^2 k''^2}{2m} - E')\delta(\frac{\hbar^2 k''^2}{2m} - E) \\
&= \int \frac{k'' m}{\hbar^2} dE'' |N_l|^2 \delta(\frac{\hbar^2 k''^2}{2m} - E')\delta(\frac{\hbar^2 k''^2}{2m} - E) \\
&= \frac{k'' m}{\hbar^2} |N_l|^2 \delta(E - E')\delta_{ll'}\delta_{mm'} \\
\rightarrow g_l &= \frac{\hbar}{\sqrt{k'' m}} \delta(E - \frac{\hbar^2 k''^2}{2m})
\end{aligned}
$$

From which we get

$$
\langle \mathbf{k} \mid E, lm, \rangle = \frac{\hbar}{\sqrt{km}} Y_l^m(\theta,\phi) \delta(E - \frac{\hbar^2 k^2}{2m}) \tag{1.14}
$$

1.3.1 $\rho_0 \rightarrow \pi\pi$

The ρ meson is spin 1 and it decays to two spin 0 pions. Suppose that the ρ is in the $l = 1, m = 1$ state, where there is some z-axis defined by something. The final state has the same angular momentum quantum numbers and the amplitude to find a π with momentum in the $\hat{\mathbf{k}}$ direction is

$$
\langle \mathbf{k} \mid E, l, m\rangle \propto Y_1^1(\hat{\mathbf{k}}) \propto \sin\theta
$$

The angular distribution of the π is

$$
|Y_1^1|^2 \sim \sin^2\theta
$$

If we imagine producing ρ in e^+e^- collisions where electrons and positrons are polarized so that $j_z = +1$ along the z-axis defined by the direction of the positron beam, then

$$
\frac{d\sigma}{d\Omega}(\theta) \propto \sin^2\theta
$$

1.3.2 Back to partial wave expansion

Substituting into Equation 1.11 we have

$$\begin{aligned}
f(\mathbf{k}',\mathbf{k}) &= -\frac{1}{4\pi}\frac{2m}{\hbar^2}8\pi^3\int dE\int dE'\sum_{l,m}\sum_{l',m'}\langle\mathbf{k}'\mid E',l',m'\rangle\langle E',l',m'\,|T_l|\,E,l,m\rangle\langle E,l,m\mid\mathbf{k}\rangle\\
&= -\frac{1}{4\pi}\frac{2m}{\hbar^2}8\pi^3\int dE\int dE'\sum_{l,m}\langle\mathbf{k}'\mid E',l,m\rangle T_l\langle E,l,m\mid\mathbf{k}\rangle\\
&= -\frac{1}{4\pi}\frac{2m}{\hbar^2}8\pi^3\int dE\int dE'\sum_{l,m}\frac{\hbar}{\sqrt{k'm}}Y_l^m(\mathbf{k}')\delta(E'-\frac{\hbar^2k'^2}{2m})T_l\frac{\hbar}{\sqrt{km}}Y_l^{m*}(\mathbf{k})\delta(E-\frac{\hbar^2k^2}{2m})\\
&= -\frac{1}{4\pi}\frac{2m}{\hbar^2}8\pi^3\sum_{l,m}\frac{\hbar^2}{km}Y_l^m(\mathbf{k}')Y_l^m(\mathbf{k})^*T_l\\
&= -\frac{4\pi^2}{k}\sum_{l,m}Y_l^m(\mathbf{k}')Y_l^m(\mathbf{k})^*T_l
\end{aligned}$$

Let $\mathbf{k}=|k|\hat{\mathbf{z}}$ so that $\theta=0,\phi=?$ and then $Y_l^m(\mathbf{k})=\sqrt{\frac{2l+1}{4\pi}}\delta_{m0}$. So only $m=0$ contributes. Then $Y_l^0(\mathbf{k}')=\sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta)$ where θ is the angle between $\mathbf{k}$ and $\mathbf{k}'$. The scattering amplitude becomes

$$f(\mathbf{k}',\mathbf{k})=-\frac{4\pi^2}{k}\sum_l\frac{2l+1}{4\pi}P_l(\theta)T_l=-\frac{\pi}{k}\sum_l(2l+1)P_l(\cos\theta)T_l \tag{1.15}$$

Define $f_l(k)=-\frac{\pi T_l(E)}{k}$ and

$$f(\mathbf{k}',\mathbf{k})=\sum_l(2l+1)P_l(\cos\theta)f_l(k) \tag{1.16}$$

$f_l(k)$ is amplitude to scatter an incident particle with angular momentum $\hbar l$ or impact parameter b such that $kb=l$. Remember that the outgoing solution to the SE far outside the range of the potential is

$$\psi^+ = \frac{1}{(2\pi)^{3/2}})\left[e^{ikz}+f(\theta)\frac{e^{ikr}}{r}\right]$$

1.3.3 Expansion of plane wave as spherical waves

The radial part of the free particle Schrodinger equation is

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[V(r) + \frac{l(l+1)\hbar^2}{2mr^2}\right]u = Eu$$

The solution to the free particle Schrodinger equation in spherical coordinates is

$$\langle \mathbf{x} \mid E, l, m\rangle = c_l j_l(kr) Y_l^m(\hat{\mathbf{r}}).$$

Next expand the plane wave as a linear combination of incoming and outgoing spherical waves.

$$\begin{aligned}
\langle \mathbf{x} \mid \mathbf{k}\rangle &= \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}} = \sum_{l,m}\int dE \langle \mathbf{x} \mid E, l, m\rangle\langle E, l, m \mid \mathbf{k}\rangle \\
&= \sum_{l,m}\int dE c_l j_l(kr) Y_l^m(\hat{\mathbf{r}})\frac{\hbar}{\sqrt{mk}}\delta(E - \frac{\hbar^2 k^2}{2m})Y_l^m(\hat{\mathbf{k}}) \\
&= \sum_l \frac{(2l+1)}{4\pi}P_k(\hat{\mathbf{k}}\cdot\hat{\mathbf{r}})\frac{\hbar}{\sqrt{mk}}c_l j_l(kr)
\end{aligned}$$

where we use the addition theorem

$$\sum_m Y_l^m(\hat{\mathbf{r}})Y_l^{m*}\hat{\mathbf{k}}) = \frac{2l+1}{4\pi}P_l(\hat{\mathbf{k}}\cdot\hat{\mathbf{r}})$$

Turns out that $c_l = \frac{i^l}{\hbar}\sqrt{\frac{2mk}{\pi}}$ so that

$$e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_l (2l+1)i^l j_l(kr)P_l(\hat{\mathbf{k}}\cdot\hat{\mathbf{r}})$$

As $r \to \infty$,

$$e^{i\mathbf{k}\cdot\mathbf{x}} \to \sum_l (2l+1)\frac{e^{i(kr))} - e^{-i(kr-(l\pi))}}{2ikr}P_l(\cos\theta) \tag{1.17}$$

1.3.4 Partial wave expansion

Now

$$\psi^+ = \frac{1}{(2\pi)^{3/2}})\left[e^{ikz} + f(\theta)\frac{e^{ikr}}{r}\right] \tag{1.18}$$

$$\psi^+ = \frac{1}{\sqrt{8\pi^3}}\sum_l i^l(2l+1)A_l(r)P_l(\cos\theta) \tag{1.19}$$

$$= \frac{1}{\sqrt{8\pi^3}}\sum_l i^l(2l+1)(c_l^1 h_l^1(r) + c_l^2 h_l^2(r))P_l(\cos\theta) \tag{1.20}$$

Remember that for large r,

$$h_l^1 \to \frac{e^{i(kr-(l\pi/2)}}{ikr}, \qquad h_l^2 \to \frac{e^{-i(kr-(l\pi/2)}}{ikr} \tag{1.21}$$

becomes, using

$$j_l(kr) \rightarrow (large\ r) \rightarrow \frac{e^{i(kr-(l\pi/2))} - e^{-i(kr-(l\pi/2))}}{2ikr}$$

Anyway, we can write, the general solution to the Schrodinger equation in the partial wave basis, far from the scattering potential as

$$\begin{aligned} \psi_{elastic} &= \psi^+ = \frac{1}{\sqrt{8\pi^3}} \sum_l i^l (2l+1) \frac{\eta_l e^{i(kr-(l\pi/2))} - c_l e^{-i(kr-(l\pi/2))}}{2ikr} \\ &= \psi^+ = \frac{1}{\sqrt{8\pi^3}} \sum_l (2l+1) \frac{\eta_l e^{i(kr))} - c_l e^{-i(kr-(l\pi))}}{2ikr} \end{aligned}$$

Then since

$$\psi_{elastic} - \frac{1}{(2\pi)^{3/2}} e^{i\mathbf{x}\cdot\mathbf{k}} = \frac{1}{(2\pi)^{3/2}} f(\theta) \frac{e^{ikr}}{r}$$

we know that $c_l = 1$ so that the ingoing wave is the same. Therefore

$$\psi_{elastic} = \psi^+ = \frac{1}{\sqrt{8\pi^3}} \frac{1}{2ikr} \sum_l i^l (2l+1)(\eta_l e^{i(kr-l\pi/2)} - e^{-i(kr-l\pi/2)})$$

Probablity conservation requres $|\eta_l| < 1$. If $|\eta_l| = 1$, the scattering is pure elastic and each partial wave gets some phase shift. If $\eta_l = 0$ the scattering for that partial wave is purely inelastic.

And we know that

$$\frac{\eta_l - 1}{2ik} = f_l$$

$$\begin{aligned} \psi^+ &= \frac{1}{(2\pi)^{3/2})} \left[\sum_l (2l+1) P_l(\cos\theta) \left(\frac{e^{ikr} - e^{-i(kr-l\pi)}}{2ikr} \right) + f(\theta) \frac{e^{ikr}}{r} \right] \\ &= \frac{1}{(2\pi)^{3/2})} \left[\sum_l (2l+1) P_l(\cos\theta) \left(\frac{e^{ikr} - e^{-i(kr-l\pi)}}{2ikr} \right) + \sum_l (2l+1) P_l(\cos\theta) f_l(k) \frac{e^{ikr}}{r} \right] \\ &= \frac{1}{(2\pi)^{3/2})} \left[\sum_l (2l+1) P_l(\cos\theta)(1 + 2ikf_l(\theta)) \left(\frac{e^{ikr}}{2ikr} \right) + \sum_l (2l+1) P_l(\cos\theta) \frac{e^{-i(kr-l\pi)}}{2ikr} \right] \end{aligned}$$

Unitarity requires that flux is conserved for each angular momentum state. Outgoing flux is no more than incoming. Therefore

$$|1 + 2ikf_l(\theta)| = |\eta_l| \leq 1 \tag{1.22}$$

and equal to one for elastic scattering.

For elastic scattering we define a phase shift

$$1 + 2ikf_l(\theta) = e^{2i\delta_l} \tag{1.23}$$

The elastic partial wave amplitude

$$f_l(\theta) = \frac{e^{2i\delta_l} - 1}{2ik} = \frac{e^{i\delta_l} \sin\delta_l}{k}$$

and

$$f(\theta) \quad = \quad \frac{1}{k}\sum_l (2l+1)e^{i\delta_l}\sin\delta_l P_l(\cos\theta) \tag{1.24}$$

The total cross section is

$$\begin{aligned}
\sigma_t &= \int d\Omega \sum_l (2l+1)^2 |f_l|^2 P_l^2(\cos\theta) && (1.25)\\
&= 4\pi \sum_l (2l+1)|f_l|^2 && (1.26)\\
&= \frac{\pi}{k^2}\sum_l (2l+1)|\eta_l - 1|^2 = \frac{\pi}{k^2}\sum_l (2l+1)(|\eta_l|^2 + 1 - 2\Re(\eta_l)) && (1.27)\\
& && (1.28)
\end{aligned}$$

If the scattering is elastic then the total cross section is

$$\begin{aligned}
\sigma_{ela} &= \frac{1}{k^2}\int d\Omega \sum_l (2l+1)^2 \sin^2\delta_l P_l^2(\cos\theta)\\
&= \frac{4\pi}{k^2}\sum_l (2l+1)\sin^2\delta_l
\end{aligned}$$

where $\eta_l = e^{2i\delta_l}$. Suppose that there is an inelastic component, so that the magnitude of the outgoing wave at momentum k in ψ_{scat} is less than the magnitude in the incoming plane wave. Then $|\eta_l| < 1$. The inelastic cross section is the piece lost from the outgoing, namely $1 - |\eta_l|^2$. Therefore

$$\sigma_{inelastic} = 4\pi \sum_l (2l+1)\frac{(1-|\eta_l|^2)}{|2ik|^2} = \frac{\pi}{k^2}(2l+1)(1-|\eta_l|^2) \tag{1.29}$$

And the optical theorem ?

$$\begin{aligned}
f(\theta) &= \sum_l (2l+1)\frac{\eta_l - 1}{2ik}P_l(\cos\theta)\\
\mathrm{Im} f(0) &= -\sum_l (2l+1)\Re\left(\frac{\eta_l - 1}{2k}\right)\\
\sigma_{tot} &= -\frac{2\pi}{k^2}\sum_l (2l+1)\Re(\eta_l - 1)
\end{aligned}$$

The inelastic cross section is the difference of the total and the elastic

$$\begin{aligned}
\sigma_{ine} &= \sigma_{tot} - \sigma_{elas}\\
&= -\frac{2\pi}{k^2}\sum_l (2l+1)\Re(\eta_l - 1) - \frac{\pi}{k^2}\sum_l (2l+1)(|\eta_l|^2 + 1 - 2\Re(\eta_l))\\
&= \frac{\pi}{k^2}\sum_l (2l+1)(1-|\eta_l|^2)
\end{aligned}$$