LECTURE 10

Single particle acceleration:
Phase stability

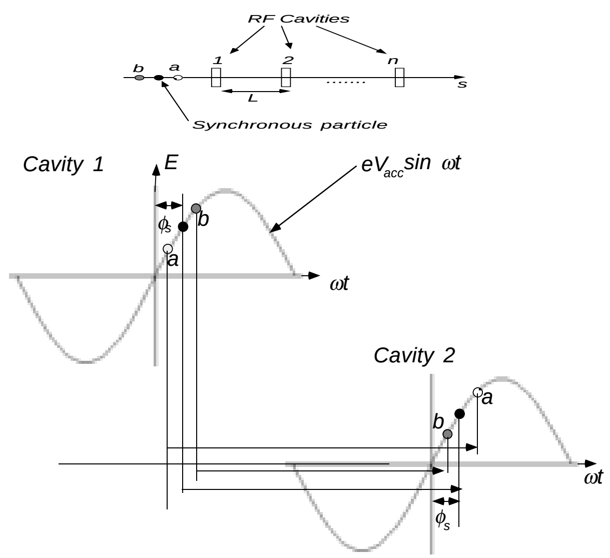
Linear Accelerator Dynamics:

Longitudinal equations of motion:
Small amplitude motion
Longitudinal emittance and adiabatic damping
Large amplitude motion

We now begin to consider the interaction between the longitudinal electric field in accelerating cavities or waveguides, and charged particles.

The most important feature of this interaction is the *principle of phase stability*.

This applies both to linacs and synchrotrons, both for standing wave and traveling wave structures.
It is this feature that allows us to accelerate simultaneously a group of particles, with a spread in energies and a spread in time: a *bunch*



Consider 3 particles entering a string of rf cavities (the reasoning is identical for a travelling wave structure). One is at the reference energy (this particle is called the *synchronous particle*); one (b) is slow, and one (a) is fast. The synchronous particle arrives at cavity 1 at time $t_s = \frac{\phi_s}{\omega}$ and gains energy $\Delta E_s = eV_{acc}\sin\phi_s$.
V_{acc} is the effective accelerating voltage (includes the transit time factor).
ϕ_s is called the *synchronous phase*.
For synchronism, the rf cavities must be spaced by $L = h\beta_s\lambda$, where h is the number of rf cycles between cavities (called the *harmonic number*), β_s is the synchronous velocity after the cavity, and λ is the rf wavelength.
The fast particle, a, arrives at $t_a < t_s$ and gains energy $\Delta E_a < \Delta E_s$.

The slow particle, b, arrives at $t_b > t_s$ and gains energy $\Delta E_b > \Delta E_s$. The synchronous particle arrives at the next cavity at the same phase ϕ_s,(this is the definition of the synchronous particle: it is in perfect synchronism with the rf). But, particle a, having gained less energy and velocity, slips later, while particle b, with a higher velocity, slips earlier.
In subsequent cavities, particles a and b will oscillate in phase about the synchronous particle. This oscillation is called a *synchrotron oscillation.*

Let's see how this works out quantitatively.

Linear Accelerator Dynamics: Longitudinal equations of motion

The synchronous particle has energy E_s, and always arrives at an rf cavity at a time $t_s = \dfrac{\phi_s}{\omega}$ relative to the rf zero-crossing.
The rf cavities are numbered by the index *n*. We'll measure the energy of non-synchronous particles relative to that of the synchronous particle; then, at cavity *n*, the non-synchronous particle's time and relative energy are

$$t_n, \quad \Delta E_n = E_n - E_{s,n}$$

in which the time is measured from the zero-crossing of the rf in cavity *n*.
The energy change between one cavity and the next is

$$E_{n+1} - E_n = \frac{dE_n}{dn} = eV_n \sin(\omega t_n), \quad \frac{dE_{s,n}}{dn} = eV_n \sin(\omega t_s) \Rightarrow$$

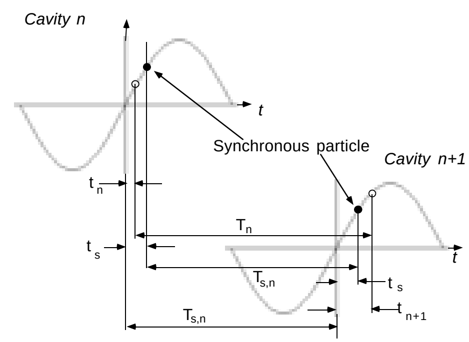
$$\frac{d}{dn}(\Delta E_n) = eV_n\left[\sin(\omega t_n) - \sin(\omega t_s)\right]$$

in which V_n is the effective accelerating voltage at cavity *n*. Note that, strictly speaking, for rf cavities, this should be a difference equation, not a differential equation. However, we'll be focusing on cases in which the energy change per cavity is a small fraction of the energy, so the use of a differential is appropriate.

How does the time t_n change from cavity to cavity?
The change in the *transit time* from one cavity to the next is due to the change in energy that has occurred as a result of the acceleration in the cavity.
Let T_n be the transit time from cavity *n* to cavity *n*+1.
Then we have

$$t_{n+1} = t_n + T_n - T_{s,n}$$

$$t_{n+1} - t_n = \frac{dt_n}{dn} = T_n - T_{s,n}$$



We want to write this in terms of the small energy difference ΔE_n, so we Taylor expand

$$T_n(E_n) = T_{s,n}(E_{s,n}) + \left.\frac{dT}{dE}\right|_{E_{s,n}} \left(E_n - E_{s,n}\right)$$

$$T_n(E_n) - T_{s,n}(E_{s,n}) = \left.\frac{dT}{dE}\right|_{E_{s,n}} \Delta E_n$$

From Lecture 6, p. 32:

$\frac{dt}{t} = \eta_C \frac{dp}{p}$, in which, for a linac, $\eta_C = -\frac{1}{\gamma^2}$

From relativistic kinematics, $\frac{dp}{p} = \frac{1}{\beta^2}\frac{dE}{E}$.

Putting these together, we have

$$\frac{dt}{t} = \frac{\eta_C}{\beta^2}\frac{dE}{E} \quad \text{so} \quad \left.\frac{dT}{dE}\right|_{E_{s,n}} = T_{s,n}\frac{\eta_C}{E_{s,n}\beta_{s,n}^2}$$

The transit time for the synchronous particle is

$$T_s = \frac{L}{\beta_s c} = \frac{h\beta_s\lambda}{\beta_s c} = \frac{h\lambda}{c}$$

in which the n subscript is understood. Then

$$\frac{dt_n}{dn} = T_n - T_s = \left.\frac{dT}{dE}\right|_{E_s} \Delta E_n = \frac{h\lambda\eta_C}{E_s\beta_s^2 c}\Delta E_n$$

The two differential equations that govern the longitudinal dynamics are then

$$\frac{dt_n}{dn} = \frac{h\lambda\eta_C}{E_s\beta_s^2 c}\Delta E_n$$

$$\frac{d}{dn}(\Delta E_n) = eV\left[\sin(\omega t_n) - \sin(\omega t_s)\right]$$

One second-order equation can be obtained by differentiating the first equation and using the second:

$$\frac{d^2 t_n}{dn^2} = \frac{h\lambda\eta_C}{E_s\beta_s^2 c}\frac{d}{dn}(\Delta E_n) + \Delta E_n \frac{h}{c}\frac{d}{dn}\left[\frac{\lambda\eta_C}{E_s\beta_s^2}\right]$$

$$= \frac{eVh\lambda\eta_C}{E_s\beta_s^2 c}\left[\sin(\omega t_n) - \sin(\omega t_s)\right] + \Delta E_n \frac{h}{c}\frac{d}{dn}\left[\frac{\lambda\eta_C}{E_s\beta_s^2}\right]$$

We now assume that the energy of the synchronous particle, and the rf wavelength, vary very slowly with n (compared to ΔE and t),

so we can ignore their derivatives. Then, we have the second order nonlinear differential equation

$$\frac{d^2 t_n}{dn^2} = \frac{eVh\lambda\eta_C}{E_s\beta_s^2 c}\left[\sin(\omega t_n) - \sin(\omega t_s)\right]$$

Small amplitude synchrotron oscillations

We're going to start by restricting ourselves to small variations in phase from the synchronous phase, to explore some features of this equation.

Let $\Delta t_n = t_n - t_s$

If $\omega\Delta t_n \ll 1$, then we can expand and approximate

$$\sin(\omega t_n)-\sin(\omega t_s)=\sin(\omega(\Delta t_n+t_s))-\sin(\omega t_s)$$
$$=\sin\omega\Delta t_n\cos\omega t_s+\cos\omega\Delta t_n\sin\omega t_s-\sin(\omega t_s)$$
$$\approx\omega\Delta t_n\cos\phi_s$$

in which $\phi_s=\omega t_s$.

This gives us a simple linear differential equation

$$\frac{d^2}{dn^2}(\Delta t_n)+(2\pi Q_s)^2\Delta t_n=0$$
$$Q_s^2=-\frac{eVh\eta_C\cos\phi_s}{2\pi E_s\beta_s^2}$$

This equation describes the small amplitude oscillations of a particle about the synchronous particle, in both energy and time, as it is accelerated in the series of rf cavities.
Q_s is called the *small amplitude synchrotron oscillation tune*. It is the number of synchrotron oscillations between rf cavities. It must be positive for stable motion. For a linac, $\eta_C=-\frac{1}{\gamma_s^2}$ and $Q_s^2=\frac{eVh\cos\phi_s}{2\pi E_s\beta_s^2\gamma_s^2}$ and so $-\frac{\pi}{2}\le\phi_s\le\frac{\pi}{2}$. We only have stable motion for this range of synchronous phase.
Provided that $Q_s^2>0$, the motion is simple harmonic:
Using $\frac{dt_n}{dn}=\frac{h\lambda\eta_C}{E_s\beta_s^2c}\Delta E_n$, we get

$$\Delta t_n=\Delta t_0\cos 2\pi Q_s n+\Delta E_0\frac{\eta_C h\lambda}{2\pi\beta_s^2E_s cQ_s}\sin 2\pi Q_s n$$

$$\Delta E_n=\Delta E_0\cos 2\pi Q_s n-\Delta t_0\frac{2\pi\beta_s^2E_s cQ_s}{\eta_C h\lambda}\sin 2\pi Q_s n$$

This can be written in the form of a matrix:

$$\begin{pmatrix}\Delta t\\ \Delta E\end{pmatrix}\Bigg|_n=\begin{pmatrix}\cos 2\pi Q_s n & \frac{\eta_C h\lambda}{2\pi\beta_s^2E_s cQ_s}\sin 2\pi Q_s n\\ -\frac{2\pi\beta_s^2E_s cQ_s}{\eta_C h\lambda}\sin 2\pi Q_s n & \cos 2\pi Q_s n\end{pmatrix}\begin{pmatrix}\Delta t\\ \Delta E\end{pmatrix}\Bigg|_0$$

which, by analogy with the transverse case, suggests the introduction of the *longitudinal Twiss parameter* β_L:

$$\beta_L=\frac{|\eta_C|h\lambda}{2\pi\beta_s^2E_s cQ_s}=\frac{\lambda}{c\beta_s}\sqrt{-\frac{\eta_C h}{2\pi eVE_s\cos\phi_s}}$$

(To keep β_L positive, we need to define it in terms of $|\eta_C|$. For $\eta_C<0$, this requires a redefinition of ΔE_n to $\Delta E_n=E_{s,n}-E_n$).

Then the longitudinal motion is

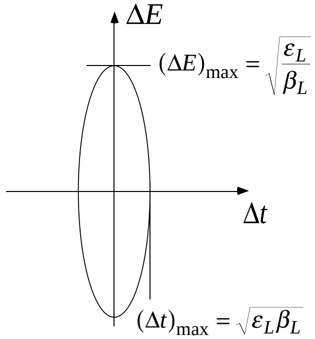
$$\begin{pmatrix}\Delta t\\ \Delta E\end{pmatrix}\Bigg|_n=\begin{pmatrix}\cos 2\pi Q_s n & \beta_L\sin 2\pi Q_s n\\ -\frac{1}{\beta_L}\sin 2\pi Q_s n & \cos 2\pi Q_s n\end{pmatrix}\begin{pmatrix}\Delta t\\ \Delta E\end{pmatrix}\Bigg|_0$$

An invariant of the motion is

$$\frac{1}{\beta_L}(\Delta t_n)^2+\beta_L(\Delta E_n)^2=\text{constant}=\varepsilon_L$$

in which ε_L is called the *longitudinal emittance*.

Longitudinal phase space is formed by the variables ΔE_n and Δt_n . In this phase space, these variables, evaluated at subsequent rf cavities, trace out an ellipse, whose area is $\pi\varepsilon_L$.



The longitudinal emittance for a beam of particles is defined in the same way as for the transverse emittance:

The *rms longitudinal emittance* is the area (divided by π) of the ellipse containing 39% of the particles.
If the distribution is Gaussian, then we have

$$\sqrt{\left\langle (\Delta t)^2 \right\rangle} = \sqrt{\beta_L \varepsilon_{L,rms}} \qquad \frac{\Delta t}{\Delta E} = \beta_L$$

$$\sqrt{\left\langle (\Delta E)^2 \right\rangle} = \sqrt{\frac{\varepsilon_{L,rms}}{\beta_L}}$$

The *rms bunch length* of this collection of particles is given by

$$\sqrt{\left\langle (\Delta s)^2 \right\rangle} = \sqrt{\left\langle (\beta_s c \Delta t)^2 \right\rangle} = \beta_s c \sqrt{\beta_L \varepsilon_{L,rms}}$$

As in transverse phase space, the *local phase space density in longitudinal phase space is constant* (Liouville's theorem).

This theorem does not hold in the presence of particle losses, dissipative processes (like scattering), or damping processes (like radiation damping or cooling).

For (ΔE, Δt) phase space, it does hold in the presence of acceleration:
The longitudinal emittance ε_L is an *adiabatic invariant:* it remains constant even if the synchronous energy, velocity and phase change, or if the rf voltage or frequency changes, as long as the

changes are slow compared to a synchrotron oscillation period. Thus, we have

$$(\Delta E)_{\max} = \sqrt{\frac{\varepsilon_L}{\beta_L}} = \left(\frac{\varepsilon_L^2 \, 2\pi mc^2 eV \beta_s^2 \gamma_s^3 \cos\phi_s}{h\lambda^2} \right)^{\frac{1}{4}}$$

$$(\Delta t)_{\max} = \sqrt{\varepsilon_L \beta_L} = \left(\frac{\varepsilon_L^2 h \lambda^2}{2\pi mc^2 eV \beta_s^2 \gamma_s^3 \cos\phi_s} \right)^{\frac{1}{4}}$$

So, as we accelerate the beam, or if we increase the rf voltage V, the energy spread $(\Delta E)_{\max} \propto \left(V\beta_s^2\gamma_s^3\right)^{\frac{1}{4}}$ increases, but the time spread $(\Delta t)_{\max} \propto \left(V\beta_s^2\gamma_s^3\right)^{-\frac{1}{4}}$ decreases. This is called *adiabatic damping* (in longitudinal phase space).

Example: Fermilab Side-coupled linac

This machine accelerates a proton beam from 116 MeV (the output of an Alvarez linac) to 400 MeV. There are about 450 cells (cavities) in about 50 m, so each cell is about 0.11 m long (the cells actually vary from about 0.08 m at the low energy end, to 0.13 m at the high energy end).

The accelerating gradient is about 8.4 MV/m; the transit time factor is about 0.85; so the acceleration per cell is about V=8.4x0.85x0.11= 0.78 MV.

The rf frequency is 805 MHz, so λ=37.2 cm. The synchronous phase is $\phi_s = 58^o$. The side-coupled cavity structure has π phase advance per cell, so h=1/2. The longitudinal emittance is $\varepsilon_{L,rms}$ = 6.4 eV-μsec.

Using these numbers, we find

Parameter	116 MeV	400 MeV	Units
β_s	0.456	0.713	
γ_s	1.12	1.42	
L	0.085	0.132	m
Q_s	0.032	0.0089	
$1/Q_s$	30.4	112.1	
β_L	$9.9\text{x}10^{-17}$	$2.79\text{x}10^{-17}$	s/eV
σ_E	0.256	0.492	MeV
σ_E/E	0.0022	0.0012	
σ_t	25	13	ps
σ_s	3.46	2.82	mm

Large amplitude synchrotron oscillations

We go back to two first-order nonlinear differential equations we obtained on p. 11:

$$\frac{dt_n}{dn} = \frac{h\lambda\eta_C}{E_s\beta_s^2 c}\Delta E_n$$

$$\frac{d}{dn}(\Delta E_n) = eV[\sin(\omega t_n) - \sin(\omega t_s)]$$

Using the chain rule, and dropping the n subscript in what follows, we can write

$$\frac{d}{d\phi}(\Delta E) = \frac{d}{dn}(\Delta E)\frac{dn}{dt}\frac{dt}{d\phi}$$

in which $\phi = \omega t$ is the phase of the particle under consideration. Then we have

$$\frac{d}{d\phi}(\Delta E) = eV[\sin\phi - \sin\phi_s]\frac{E_s\beta_s^2 c}{\omega\lambda h\eta_C\Delta E}$$

$$= -[\sin\phi - \sin\phi_s]\frac{1}{\cos\phi_s\omega^2\beta_L^2\Delta E}$$

in which $\beta_L^2 = -\frac{h\eta_C\lambda^2}{2\pi\cos\phi_s E_s\beta_s^2 eVc^2}$ has been used. So

$$\Delta E d(\Delta E) = -\frac{[\sin\phi - \sin\phi_s]}{\cos\phi_s\omega^2\beta_L^2}d\phi$$

Assuming that ϕ_s, ω and β_L are approximately constant during a synchrotron oscillation, we can integrate both sides to give

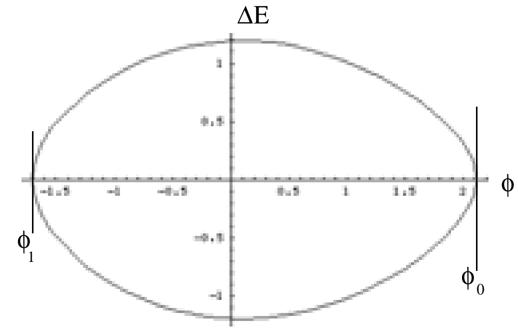
$$\frac{1}{2}(\Delta E)^2 = \frac{[\cos\phi + \phi\sin\phi_s] - [\cos\phi_0 + \phi_0\sin\phi_s]}{\cos\phi_s \omega^2 \beta_L^2}$$

in which ϕ_0 is the phase for which ΔE=0.
This can be written as

$$\omega\beta_L(\Delta E)^2 - \frac{2}{\beta_L\omega\cos\phi_s}(\cos\phi + \phi\sin\phi_s)$$
$$= -\frac{2}{\beta_L\omega\cos\phi_s}(\cos\phi_0 + \phi_0\sin\phi_s)$$

This equation gives the curve in (ΔE, ϕ) phase space corresponding to a large amplitude synchrotron oscillation. This curve is sometimes called a *phase space trajectory*.

What do these curves look like? Here's a typical one

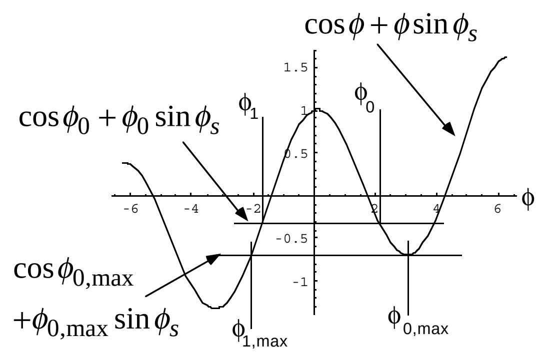


The curve crosses the line ΔE=0 at the two points, ϕ_0 and ϕ_1, where ϕ_1 is given in terms of ϕ_0 by the equation

$$\cos\phi_1 + \phi_1\sin\phi_s = \cos\phi_0 + \phi_0\sin\phi_s$$

These two values of ϕ are the *bounds* of the motion, for this particular phase space trajectory.
Phase space trajectories corresponding to larger values of ϕ_0 are possible, up to a maximum $\phi_{0,max}$

To see this,
we plot $\cos\phi + \phi\sin\phi_s$ (for ϕ_s = 0.1).



$\phi_{0,max}$ and $\phi_{1,max}$, correspond to the maximum extent of bounded motion possible. For larger values of ϕ, the motion is not bounded.

$\phi_{0,max}$ occurs at a minimum in the function $\cos\phi + \phi\sin\phi_s$.

By differentiating the function, we see that this occurs at

$$\phi_{0,\max} = \pi - \phi_s$$

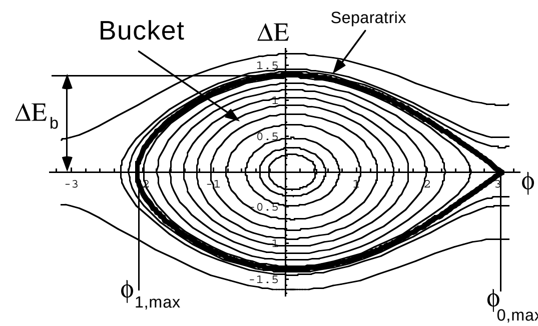
The other bound to the motion may be found from

$$\cos\phi_{1,\max} + \phi_{1,\max}\sin\phi_s = \cos(\pi-\phi_s) + (\pi-\phi_s)\sin\phi_s$$
$$= -\cos\phi_s + (\pi-\phi_s)\sin\phi_s$$

The phase space trajectory corresponding to the maximum bounded motion

$$\omega\beta_L(\Delta E)^2 - \frac{2}{\beta_L\omega\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right)$$
$$= \frac{2}{\beta_L\omega\cos\phi_s}\left(\cos\phi_s - (\pi-\phi_s)\sin\phi_s\right)$$

is called the *separatrix*: it separates bounded from unbounded motion



The synchrotron tune decreases as the oscillation amplitude increases; on the separatrix, the tune is zero, and the period is infinite. The area in phase space within the separatrix is called the *bucket*.

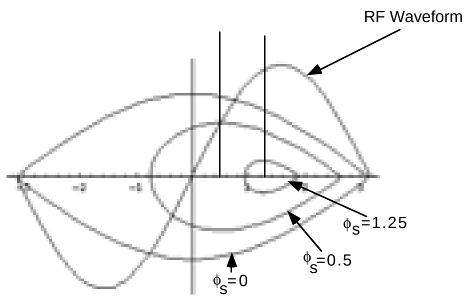
The phase space area occupied by the beam (the longitudinal emittance) must be inside the bucket (typically, well inside: it would correspond to one of the small ellipses in the figure above.) The "height" of the bucket, ΔE_b, determines the *energy acceptance* of the accelerator. This is given by setting $\phi=\phi_s$, and $\Delta E=\Delta E_b$ in the separatrix equation: the result is

$$\Delta E_b = \frac{2\sqrt{1-\left(\frac{\pi}{2}-\phi_s\right)\tan\phi_s}}{\omega\beta_L}$$

The bucket represents the maximum stable area in phase space.

For zero synchronous phase (no acceleration), the bucket spans the whole range of ϕ from -π to π.

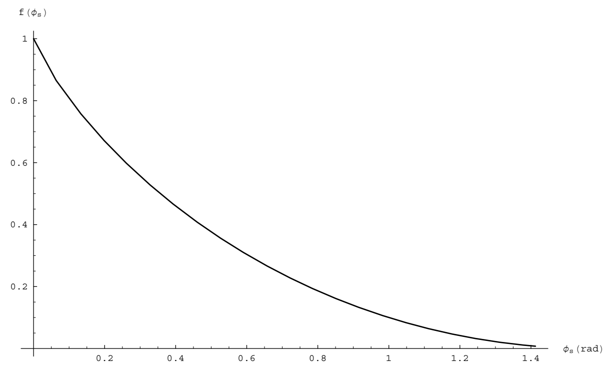
As the synchronous phase increases, the size of the bucket shrinks, both in phase and in energy.



The *bucket area*, the area within the separatrix (in ΔE, Δt phase space), can be found by integrating over the bucket; the result is

$$\frac{A_b}{\pi} = \frac{16}{\pi\omega^2\beta_L} f(\phi_s)$$

in which the function $f(\phi_s)$ is:



For good performance, the longitudinal emittance of the beam should be much smaller than the bucket area/π.

Example: Fermilab proton linac again:

ϕ_s=58°, $\varepsilon_{L,rms}$=6.4 eV-μs

Parameter	116 MeV	400 MeV	Units
β_s	0.456	0.713	
β_L	9.9×10^{-17}	2.79×10^{-17}	s/eV
$\phi_{1,max}$	25	25	degrees
$\phi_{2,max}$	122	122	degrees
ΔE_b	1.3	4.8	MeV
σ_E	0.256	0.492	MeV