LECTURE 11

Linear Accelerator Dynamics:
Electron Linacs
Prebunching

Longitudinal dynamics in synchrotrons
Acceleration
Matching and filamentation
Longitudinal "gymnastics":
Debunching and Bunch rotation

Synchrotron radiation: introduction

Linear Accelerator Dynamics: Electron Linacs

In electron linacs, γ_s gets very large rapidly. From the adiabatic damping relations, it would appear that ΔE would get very large, and Δt would get very small. This does not happen, because the synchrotron frequency goes to zero for large γ_s. In electron linacs, the synchrotron motion becomes "frozen" very rapidly as the beam is accelerated. Particles remain at the same values of phase and energy. The time spread Δt and energy spread ΔE remain essentially constant, and the relative energy spread $\Delta E/E$ decreases as $1/E$.

What happens in the regime before the beam becomes relativistic? To address this question, we must go back to the original equations of motion: we can no longer treat ΔE as a small quantity, since a particle's energy will be changing by large fractions of itself in this regime. We'll consider the case of an electron travelling wave linac, and write the equations in terms of the continuous variable *s*, corresponding to the continuous acceleration in a travelling wave linac.

Return to Lecture 10, p. 8:

$$t_{n+1} - t_n = L\frac{dt}{ds} = T(s) - T_p(s)$$

$$T(s) = \frac{L}{\beta c} \Rightarrow \frac{dt(s)}{ds} = \frac{1}{c}\left(\frac{1}{\beta} - \frac{1}{\beta_p}\right)$$

In this equation, $T_p(s)$ is the transit time, and β_p is the (constant) phase velocity, of the travelling electromagnetic wave.

The energy equation, from Lecture 10, p. 7, becomes

$$\frac{dE(s)}{ds} = eE_0 \sin \omega t(s)$$

We'd like to get a relation between E and $\phi = \omega t$, which will represent the motion in longitudinal phase space.

Using the chain rule

$$\frac{dE}{d\phi} = \frac{dE}{ds}\frac{ds}{dt}\frac{dt}{d\phi}$$

we get

$$\frac{dE}{d\phi} = \frac{ceE_0}{\omega}\sin\phi\left(\frac{1}{\beta} - \frac{1}{\beta_p}\right)^{-1}$$

$$dE\left(\frac{1}{\beta} - \frac{1}{\beta_p}\right) = \frac{ceE_0}{\omega}\sin\phi d\phi$$

This can be written in terms of perfect differentials as

$$d\left[E\left(\beta - \frac{1}{\beta_p}\right)\right] = \frac{ceE_0}{\omega}d[-\cos\phi]$$

which integrates to

$$\frac{\omega}{ceE_0}E\left(\beta - \frac{1}{\beta_p}\right) = \cos\phi_0 - \cos\phi$$

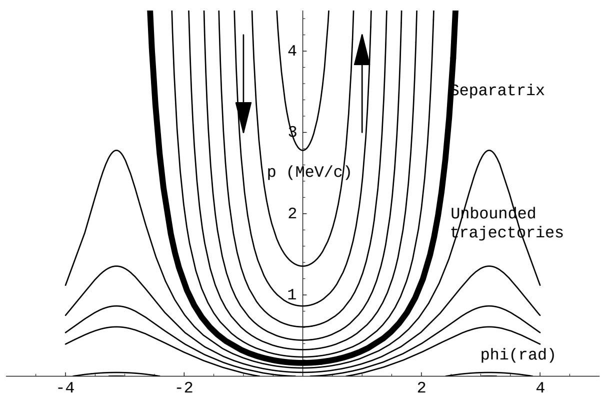
in which ϕ_0 is the phase for which $\beta=\beta_p$. For an electron linac, the phase velocity of the travelling wave is chosen to be $v_p=c$, so $\beta_p=1$ and we have

$$\frac{\omega}{ceE_0}E(\beta - 1) = \cos\phi_\infty - \cos\phi$$

as the equation of the trajectories in longitudinal phase space. ϕ_0 has been renamed ϕ_∞, the *asymptotic phase*. In terms of the particle momentum p, and the rf wavelength $\lambda = \frac{2\pi c}{\omega}$, we have

$$\frac{2\pi c}{eE_0\lambda}\left(\sqrt{m^2c^2 + p^2} - p\right) = \cos\phi - \cos\phi_\infty$$

The following figure shows the phase space trajectories for the case of the CESR Linac, for which λ=10.5 cm, E_0 =11 MV/m



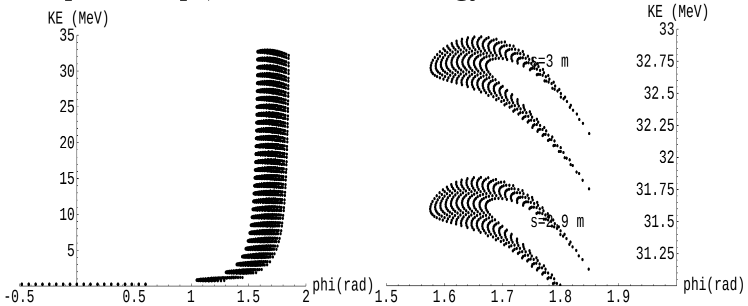
The separatrix corresponds to the trajectory with $\cos\phi_\infty = \pm 1$. The minimum injection energy on this trajectory is for ϕ=0, for which

$$\sqrt{m^2c^2 + p_{\min}^2} - p_{\min} = \frac{eE_0\lambda}{\pi c}$$

For the CESR linac case, p_{min}=164 keV/c (kinetic energy about 26 keV). Ideally, injection should occur at ϕ=0, but at an energy well above the minimum, close to the trajectory with an asymptotic phase of 90°.

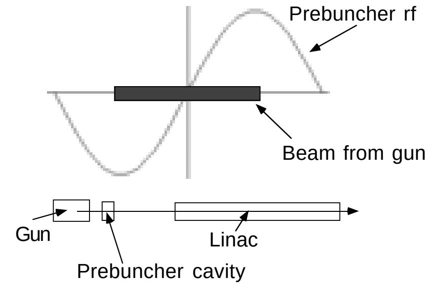
This provides maximum acceleration and minimum phase spread at the output.

The figures below show the evolution of a series of points in phase space in the beginning of the CESR linac. The points are at successive values of s, from 0 to 3 m, in steps of 0.1 m. The input phase space is a uniform distribution in ϕ from -34^{o} to 34^{o} (full time spread 66 ps) and in kinetic energy from 225 to 275 keV.

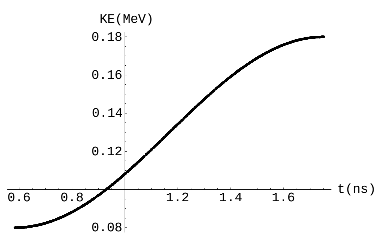


The output phase space has a full energy spread of about 1 MeV, and a full phase spread of about 15^{o} (14 ps).

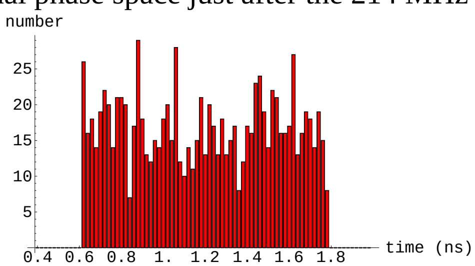
Note that the phase spread about ϕ=0 at injection is manifested as an energy spread at the output. Consequently, it's important to have a minimum phase spread at injection. Very short pulses can be obtained from photoinjectors, but thermionic DC guns with pulsed grids give nanosecond time scale pulses. A *prebuncher* is usually required to reduce the time spread further.



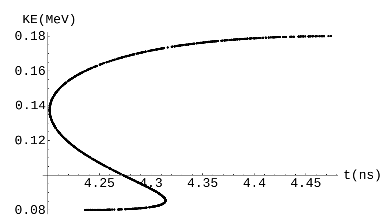
The beam from the gun is injected into the prebuncher cavity so that the early particles are decelerated, and the late ones accelerated.



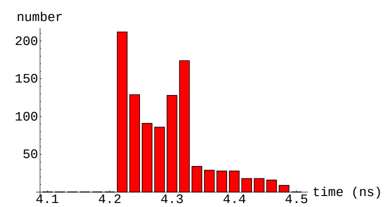
Longitudinal phase space just after the 214 MHz prebuncher



Projection on the time axis



Longitudinal phase space just before the linac



Projection on the time axis

Longitudinal dynamics in synchrotrons

The longitudinal dynamics discussion in lecture 11, which was given for (non-relativistic) linacs, applies directly, as long as the appropriate value for η_C is used. For a synchrotron, we must use the result from lecture 6, p. 32:

$$\eta_C = \alpha_C - \frac{1}{\gamma^2} = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2},$$

where $\alpha_C = \frac{1}{C}\oint_C \frac{\eta(s)}{\rho(s)} ds = \frac{1}{\gamma_t^2}$ is the momentum compaction factor.

In contrast to linacs, η_C can be positive, and in fact it can change from negative to positive during acceleration if the machine crosses *transition*, i.e., the γ of the beam goes from below γ_t to

above γ_t. What happens at *transition crossing* is a special topic to be discussed later.

Above transition, *the condition on ϕ_s for phase stability changes*: Since we always need to have

$$Q_s^2 = -\frac{eVh\eta_C\cos\phi_s}{2\pi E_s\beta_s^2} > 0,$$

if η_C is positive, cos ϕ_s must be negative, i.e., $\frac{\pi}{2} \le \phi_s \le \frac{3\pi}{2}$.

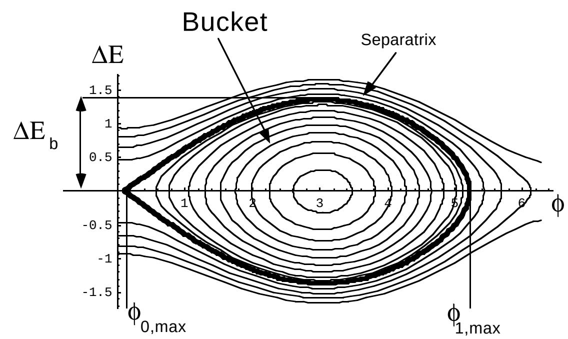
Note that above transition, higher energy particle take more time to go around the machine than lower energy particles. This is sometimes referred to as the "negative mass" effect: the revolution time increases for faster particles. We will see later that this can, in some cases, lead to a form of unstable motion.

Typically more than 1 rf cavity is present in a synchrotron. Nevertheless, the *harmonic number h* in a synchrotron is taken to be the number of rf cycles per *revolution* (rather than per cavity). Hence, for example, in our matrix equation

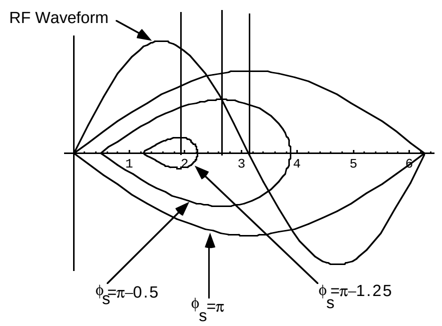
$$\begin{pmatrix}\Delta t \\ \Delta E\end{pmatrix}\Bigg|_n = \begin{pmatrix}\cos 2\pi Q_s n & \beta_L \sin 2\pi Q_s n \\ -\frac{1}{\beta_L}\sin 2\pi Q_s n & \cos 2\pi Q_s n\end{pmatrix}\begin{pmatrix}\Delta t \\ \Delta E\end{pmatrix}\Bigg|_0$$

n refers to the *turn number*, and the *V* in $Q_s^2 = -\frac{eVh\eta_C\cos\phi_s}{2\pi E_s\beta_s^2}$, and in the expression for β_L, refers to the total voltage per *turn* (summed over all cavities).

The shape of the rf buckets are "reversed" above transition (the pointy end is at small values of ϕ):



There are *h* such buckets around the machine, and there may be *h* synchronous particles at the center of each one. Thus the machine can contain at most *h* bunches.



The rf bucket corresponding to $\phi_s=\pi$, for which there is no acceleration, is called a *stationary bucket*.

Acceleration

In a synchrotron, the magnetic field is ramped as the beam energy increases, to maintain a constant orbit radius. The synchronism conditions for the synchronous particle are a condition on the rf frequency, and a relation between the rf voltage and the field ramp rate.

The rate of energy gain for the synchronous particle is

$$\frac{dE_s}{dt} = f_s eV \sin\phi_s$$

in which $f_s = \frac{\beta_s c}{C} = \frac{c}{h\lambda}$ is the revolution frequency of the synchronous particle, and C is the machine circumference. The rf wavelength must satisfy the condition

$$\lambda = \frac{C}{h\beta_s}$$

The rate of momentum change for the synchronous particle is

$$\frac{dp_s}{dt} = eE_0 \sin\phi_s = \frac{eV}{C}\sin\phi_s$$

Using $p_s = e\rho B_0$, we get

$$\frac{dB_0}{dt} = \frac{V\sin\phi_s}{\rho C}$$

This stipulates how $V\sin\phi_s$ must change during acceleration. If the magnetic field has a uniform ramp rate, with $\frac{dB_0}{dt} = \text{constant}$, then $V\sin\phi_s$ can remain constant during acceleration.

If (as is typically the case in rapid cycling synchrotrons with resonant magnet systems, like the Cornell synchrotron and the Fermilab Booster), the magnetic field varies in time like

$$B_0(t) = B_{\min} + \frac{(B_{\max} - B_{\min})}{2}(1 - \cos 2\pi f_c t)$$

in which f_c is the machine cycle frequency (e.g., 60 Hz), then synchronism requires that $V\sin\phi_s$ have a sinusoidal time dependence. Specifically, we have

$$eV\sin\phi_s = C\pi f_c (p_{\max} - p_{\min})\sin 2\pi f_c t$$

in which p_{max} and p_{min} are the maximum and minimum values of the synchronous momentum during acceleration.

Example: the Cornell synchrotron

This machine accelerates a 300 MeV electron beam to 5200 MeV in about 8 msec, corresponding to a cycle frequency of 60 Hz. It uses a fixed-frequency 714 MHz rf system, operating at a harmonic number of *h*=1800, corresponding to a circumference of about 756 m. The momentum compaction factor is α_C =0.0096.

The synchronous phase is roughly constant during acceleration, at about 150°. The rf voltage per turn has a roughly sinusoidal dependence on time, with *V*= 1 MV at the beginning and end of the cycle. In mid-cycle, at the maximum value of *dp/dt*, the required voltage, from the above equation, works out to about 4.7 MV. The injected longitudinal emittance is ε_L = 6.6 eV-μs. Synchrotron radiation does not play much of a role in the dynamics. Using these numbers, we find the following table for the parameters of the longitudinal motion:

Parameter	300 MeV	2500 MeV	5200 MeV	Units
V	1	4.7	1	MeV
γ_s	588	4893	9785	
Q_s	0.089	.0666	.021	
$1/Q_s$	11.25	15.1	45.8	
β_L	144	23	35	ps/MeV
σ_E	0.214	0.534	0.432	MeV
σ_E/E	713	213	86	$x10^{-6}$
σ_t	30	12.3	15	ps
σ_s	9.3	3.7	4.6	mm
ΔE_b	1.9	12.1	7.9	MeV
A_b/π	702	4376	2867	μs-eV

Matching and filamentation

Beam transfer from one synchrotron to another, or from a linac into a synchrotron, is often done “bucket-to-bucket”: the rf systems of the two machines are phase-locked, and the bunches are transferred directly from the buckets of one machine into the buckets of the other.

This process can be quite efficient. However, growth of the longitudinal emittance will occur unless (i) the injected beam hits the middle of the bucket in the receiving machine, and (ii) the two machines are longitudinally matched. By this, it is meant that they have the same value of the longitudinal beta function β_L, which determines the aspect ratio of the longitudinal ellipse in phase space.

If the injected beam misses the bucket center, or the values of β_L are unequal, then the injected beam will rotate in the bucket after injection. This rotation, coupled with the nonlinear phase space trajectories, results in *filamentation* and an effective growth in the emittance.

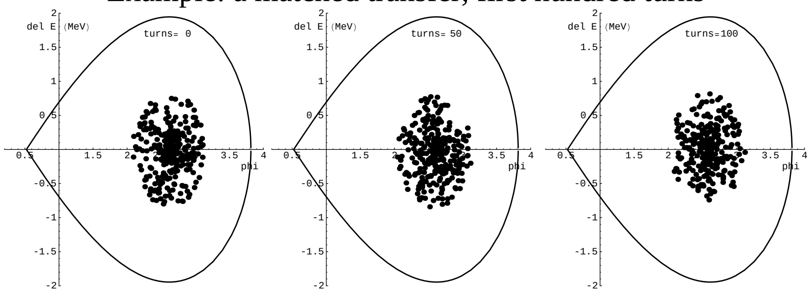
The following plots illustrate this. They were made by solving numerically the exact differential equations of motion in the form:

$$\frac{d\phi}{dn} = 2\pi\beta_L Q_s \Delta E$$

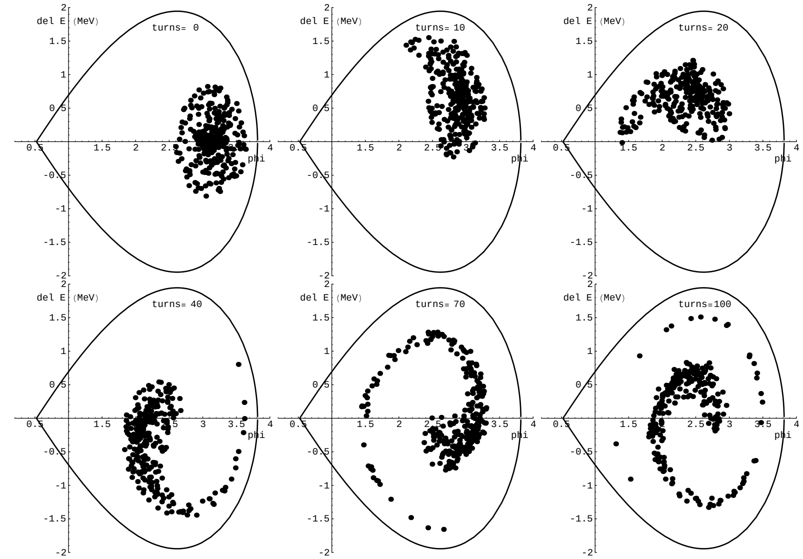
$$\frac{d}{dn}(\Delta E) = eV(\sin\phi - \sin\phi_s)$$

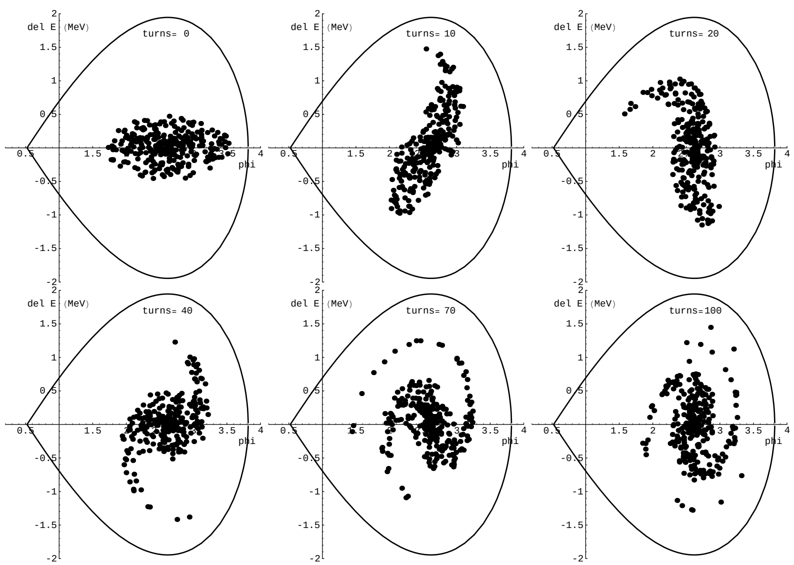
with the longitudinal parameters for the Cornell synchrotron at injection, and for a beam with ε_L = 100 eV-μsec.

Example: a matched transfer, first hundred turns



After this matched transfer, the emittance does not grow.
Mismatched transfers: phase error of 60^{o}, first hundred turns (next page); No phase error, β_L error of factor of 3 (following page):



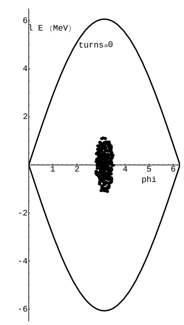


<u>Longitudinal "gymnastics": Debunching and Bunch rotation</u>

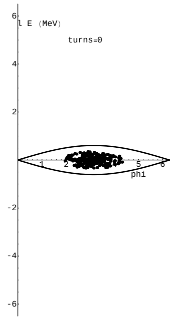
These are processes by which the time spread and energy spread of a bunch can be manipulated.

They are illustrated in the following series of figures.

We start with a matched bunch in a stationary bucket. The rf voltage is V_0.



The rf voltage is adiabatically reduced to a small value V_{min}, over many synchrotron oscillation periods. The bunch remains matched to the bucket; the energy spread goes down like $V^{0.25}$, and the time spread (bunch length) goes up like $V^{-0.25}$. The product must remain the same by Liouville's theorem.

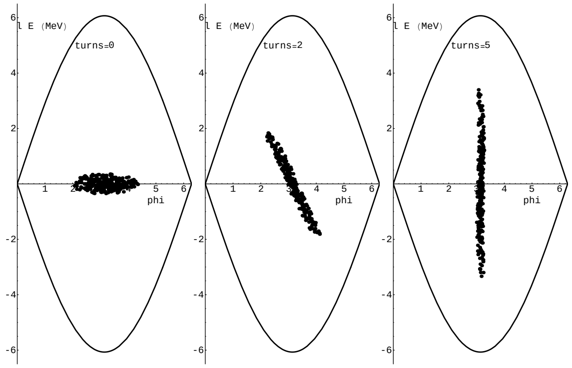


If this process is continued, with the rf voltage eventually being turned off, the beam is said to be *debunched*. The beam is then distributed around the whole circumference, rather than being contained in bunches by rf buckets. The energy spread is reduced in the debunching process: if the process is truly adiabatic, the longitudinal emittance is conserved, and the final (full) energy spread is just

$$\Delta E_{debunched,\,full} = f_s \varepsilon_L \quad \text{(per bunch)}$$

where f_s is the revolution frequency of the synchronous particle. In the debunched state, no rf voltage is applied to the beam. At any significant energy, it is impossible to debunch an electron beam, as energy must be always be supplied to make for the synchrotron radiation.

Suppose that, after the rf voltage is reduced substantially but not brought to zero, the rf voltage is suddenly snapped back up to the original value, in a time much shorter than a synchrotron oscillation. The mismatched bunch rotates in the bucket:



After a quarter of a synchrotron period, the bunch is very narrow, although with a large energy spread. If it is extracted from the machine at this point, a beam with a very narrow time spread can be provided.

The typical use of such a beam is for high intensity production of secondary particles. The narrow time spread is carried over to the time distribution of the secondary particles, and enhances the longitudinal density of the secondary beam.

Exercise: show that the ideal overall bunch length reduction factor in this process is $\left(\dfrac{V_{\min}}{V_0}\right)^{0.25}$

Synchrotron radiation

Up to this point, we have not considered the fact that charged particles radiate energy when they are accelerated. This will turn out not to be very important for linacs *at any energy*, or for synchrotrons in which γ is not >>1.

However, when γ>>1 in a synchrotron, (such as in virtually all electron synchrotrons, and in very high-energy proton synchrotrons), there will be considerable radiation, which will play a very important role in the particle dynamics.

Two results from classical E&M will form the starting point of our discussion:

1. The Lienard formula for the total power radiated by an accelerated charged particle of charge *e*, having acceleration $\vec{a}$, velocity $\vec{\beta}c$, and energy $m_0\gamma c^2$

$$P = \frac{1}{6\pi\varepsilon_0}\frac{e^2}{c^3}\gamma^6\left[a^2 - \left|\vec{\beta}\times\vec{a}\right|^2\right]$$

2. The acceleration experienced by such a particle in the fields $\vec{E}$ and $\vec{B}$

$$\vec{a} = \frac{e}{m_0\gamma}\left[\vec{E} - \vec{\beta}\left(\vec{\beta}\bullet\vec{E}\right) + c\vec{\beta}\times\vec{B}\right]$$

Let us apply these equations to the reference particle in an accelerator.

On the reference orbit, $\vec{E}$ and $\vec{\beta}$ are always parallel, so the acceleration becomes

$$\vec{a} = \frac{e}{m_0\gamma}\left[\frac{\vec{E}}{\gamma^2} + c\vec{\beta}\times\vec{B}\right]$$

Let us consider first the acceleration due to $\vec{E}$, and the resulting radiation power:

$$\vec{a}_E = \frac{e\vec{E}}{m_0\gamma^3}; \quad \vec{\beta}\times\vec{a}_E = 0$$

$$P_E = \frac{1}{6\pi\varepsilon_0}\frac{e^2}{c^3}\gamma^6\left[\frac{e\vec{E}}{m_0\gamma^3}\right]^2 = \frac{1}{6\pi\varepsilon_0}\frac{e^4E^2}{m_0^2c^3}$$

For linacs, in which the magnetic field on the reference orbit is zero, this will give the radiated power. Note that the power is independent of the energy. In terms of the classical radius $r_0 = \dfrac{e^2}{4\pi\varepsilon_0 m_0 c^2}$, the power is $P_E = \dfrac{2}{3}\dfrac{c}{r_0}\dfrac{(eEr_0)^2}{m_0c^2}$. For example, for electrons, $r_0 = 2.82\text{x}10^{-15}$ m, and we have, for a large field E=100 MV/m, $P_E = 1.7\ 10^{-15}$ W. For a beam of 10^{12} electrons, this is only about 1.7 mW.

Now consider the acceleration due to $\vec{B}$ (the centripetal acceleration in synchrotrons). We have

$$\vec{a}_B = \frac{ec}{m_0\gamma}\left[\vec{\beta}\times\vec{B}\right] = \frac{ec\beta B}{m_0\gamma}\hat{a}_B$$

$$\left|\vec{\beta}\times\vec{a}_B\right| = \beta a_B$$

since, on the reference orbit, $\vec{\beta}_s$ and $\vec{B}$ are perpendicular. So

$$P_B = \frac{1}{6\pi\varepsilon_0}\frac{e^2}{c^3}\gamma^6\left[a_B^2 - \left|\vec{\beta}\times\vec{a}_B\right|^2\right]$$

$$= \frac{1}{6\pi\varepsilon_0}\frac{e^2}{c^3}\gamma^4 a_B^2 = \frac{1}{6\pi\varepsilon_0}\frac{e^4 B^2}{m_0^2 c}(\beta\gamma)^2$$

For synchrotrons, this will give the radiated power: that due to the electric fields is negligible, as we've seen above. This type of radiation, associated with the centripetal acceleration a charged particle, is called *synchrotron radiation*. Note that the power

depends quadratically on the energy for relativistic particles. In terms of the classical radius, it is

$$P_B = \frac{2}{3}\frac{c}{r_0}\frac{(eBcr_0)^2}{m_0c^2}(\beta\gamma)^2$$

In practical units, for electrons, we have

$$P_B = 1.59\times 10^{-14}(B[\mathrm{T}])^2(\beta\gamma)^2 \quad \mathrm{W}$$

radiated per electron.

Note the strong energy dependence of this power.

For a 5 GeV beam of 10^{12} electrons in a 1 T field, the power radiated is 1.59 MW. In addition to being a dominant concern for the rf system, which must supply this power, the radiation has a substantial impact on the beam dynamics, which we'll discuss next.