LECTURE 12

Synchrotron radiation: Longitudinal effects

Damping of synchrotron oscillations
Features of synchrotron radiation

Equations for the damping and quantum excitation of synchrotron oscillations:

Energy damping time and equilibrium energy spread

Damping of synchrotron oscillations

The power radiated by a particle due to synchrotron radiation results in a *damping* of synchrotron oscillations.

Why?

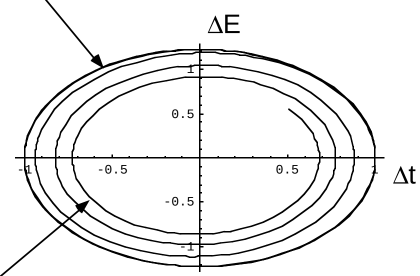
Power radiated by a particle:

$$P = \frac{1}{6\pi\varepsilon_0}\frac{e^4B^2}{m_0^2c}\gamma^2$$

(We take β=1 in what follows, as we will be dealing exclusively with relativistic particles)

Consider a particle on an elliptical trajectory in longitudinal phase space. When it is in a region of positive ΔE (energy greater than E_s), it radiates more than the synchronous particle and ΔE gets

smaller. When it is in a region of negative ΔE (energy greater than E_s), it radiates less than the synchronous particle and ΔE still gets smaller. So the trajectory spirals in toward the origin:



The energy spread and bunch length are thus damped, and decrease with time. There is a limit to this process, however. To understand this limit, we must recognize that the radiation emitted by the particle is emitted in the form of discrete energy quanta (photons).

The emission process is quantum mechanical, and hence has a random character. Statistical fluctuations in the number of photons emitted will cause fluctuations in the energy of the particle.

These fluctuations increase the energy spread of the beam and establish the limit to which it will ultimately damp.

Let's try to quantify the effects of synchrotron radiation on the energy spread. We will find *two important results*, which can be stated (approximately) very simply:

1. For a separated function lattice, the energy damping time is approximately equal to the time required for a particle to radiate all its energy. If T_s =revolution period, and U_s = energy loss per turn, then the energy damping time is

$$\tau_{\Delta E} \approx T_s \frac{E_s}{U_s}$$

2. The equilibrium rms energy spread is approximately the rms photon energy times the square root of the number of photons emitted during one damping time. If $\dot{N}$=photon emission rate, and $\sqrt{\langle u^2 \rangle}$ is the rms photon energy, then the rms energy spread is

$$\sigma_{\Delta E} \approx \sqrt{\dot{N} \tau_{\Delta E} \langle u^2 \rangle}$$

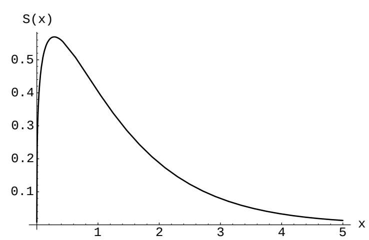
Before we can see where these results come from, however, we'll need to introduce some information from electromagnetic theory.

Features of synchrotron radiation

P=the total power radiated by an electron .The *power spectrum* of the radiation is

$$\frac{dP}{d\omega} = \frac{P}{\omega_c} S\left(\frac{\omega}{\omega_c}\right)$$

in which $\omega_c = \frac{3}{2}\frac{\gamma^3 c}{\rho}$ is called the *critical frequency*. $\rho = \frac{p}{eB}$ is the bending radius of the electron. The function $S(x)$, called the normalized spectrum, is shown in the next figure.



$$S(x) = \frac{9\sqrt{3}}{8\pi} x \int_x^\infty K_{5/3}(u)du$$

This energy is radiated in the form of photons, each of energy $u = \hbar\omega$. Thus, the number of photons radiated per second, in the energy interval *du*, is

$$\dot{n}(u)du = \frac{dP}{d\omega}\frac{d\omega}{\hbar\omega}$$

So the photon rate spectrum (photons per unit energy per second) is

$$\dot{n}(u) = \frac{P}{u_c^2}\frac{u_c}{u} S\left(\frac{u}{u_c}\right)$$

in which $u_c = \hbar\omega_c$ is the critical energy.

The total number of photons emitted per second is

$$\dot{N} = \int_0^\infty \dot{n}(u)du = \frac{15\sqrt{3}}{8}\frac{P}{u_c}$$

The mean photon energy is

$$\langle u \rangle = \frac{1}{\dot{N}} \int_0^\infty u\dot{n}(u)du = \frac{P}{\dot{N}} = \frac{8}{15\sqrt{3}} u_c$$

and the mean square energy is

$$\langle u^2\rangle = \frac{1}{\dot N}\int_0^\infty u^2 \dot n(u)du = \frac{11}{27}u_c^2$$

For the synchronous particle, the energy loss *per turn* is

$$U_s = \oint_{turn} Pdt = \oint_{turn} \frac{e^4}{6\pi\varepsilon_0 m_0^2 c} B_0^2\gamma_s^2 dt = \frac{e^4\gamma_s^2}{6\pi\varepsilon_0 m_0^2 c}\int_0^C dsB_0^2 \frac{dt}{ds}$$

$$= \frac{e^4\gamma_s^2}{6\pi\varepsilon_0}\left(\frac{p_s}{m_0 ce}\right)^2\int_0^C \frac{ds}{\rho^2} = \frac{e^2\gamma_s^4}{6\pi\varepsilon_0}\int_0^C \frac{ds}{\rho^2}$$

This energy loss must be restored by the rf system, in order to keep the synchronous particle at a constant energy. Thus, assuming no acceleration, as in a storage ring, we must have

$$eV\sin\phi_s = U_s$$

For an *isomagnetic* lattice (one for which all the dipoles have the same bend radius ρ), $\int_0^C \frac{ds}{\rho^2} = \frac{2\pi}{\rho}$. In this case,

$$U_s = \frac{e^2\gamma_s^4}{3\varepsilon_0\rho} = \frac{4\pi}{3}\gamma_s^4 m_0 c^2 \frac{r_0}{\rho}.$$

Example: synchrotron radiation in CESR

In practical units, for electrons, we have (see Lecture 1, p. 24):

$$U_s[\text{MeV}] = 0.0885\frac{E^4[\text{GeV}]}{\rho[\text{m}]}$$

For CESR, using E=5.29 GeV and ρ=98 m (arc dipoles), we get U_s=0.71 MeV.

(CESR is not isomagnetic; the energy loss is actually larger than this.)

The critical photon energy is

$$u_c = \hbar\omega_c = \frac{3}{2}\frac{\hbar\gamma^3 c}{\rho}$$

In practical units, for electrons

$$u_c[\text{keV}] = 2.218\frac{E^3[\text{GeV}]}{\rho[\text{m}]}$$

For CESR, this gives u_c=3.18 keV. Using the other equations above, we find, for the synchronous particle,

$$\langle u\rangle = u_c\frac{8}{15\sqrt{3}} = 0.98 \text{ keV}$$

$$\sqrt{\langle u^2\rangle} = u_c\sqrt{\frac{11}{27}} = 2.0 \text{ keV}$$

The number of photons radiated in one turn is

$$N = \oint_{turn} \dot N dt = \frac{15\sqrt{3}}{8}\frac{U_s}{u_c} = 721$$

There are about 100 dipoles in CESR, so in each magnet, an electron radiates only about 7 photons. It is thus not surprising, if synchrotron radiation plays an important role in the beam dynamics, that fluctuations due to photon statistics will have to be included.

Rough estimates of the damping time and equilibrium energy spread:

Number of turns for an electron to radiate all its energy$=\frac{E_s}{U_s}\approx 7300$

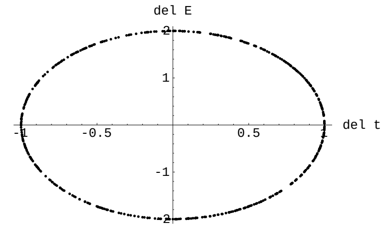
damping time~rev period x 7300=2.6x10^{-6}x7300=19 ms

energy spread~$\sqrt{7300\times 720}\times 2$ keV~5 MeV

Let's now see how to get more accurate estimates for these quantities.

Equations for the damping and quantum excitation of synchrotron oscillations:

Suppose that we have a collection of particles in longitudinal phase space, all having the same value of the longitudinal emittance, but distributed randomly around the ellipse:



In what follows, we will let *ε*=*ΔE*, to simplify the notation. For the *i*th particle, the dependence of *Δt* and *ε* on the turn number *n* can be written as

$$\varepsilon_i = \varepsilon_0 \sin(2\pi Q_s n + \psi_i)$$

$$\Delta t_i = -\beta_L \varepsilon_0 \cos(2\pi Q_s n + \psi_i)$$

The square of the amplitude of the energy oscillation is an invariant of the motion:

$$\frac{1}{\beta_L^2}(\Delta t_i)^2 + \varepsilon_i^2 = \varepsilon_o^2$$

It won't be an invariant once we allow synchrotron radiation, which is a dissipative process. Let particle *i* emit synchrotron radiation in the form of a photon of energy u_i. This corresponds to

$$\varepsilon_i \to \varepsilon_i - u_i$$

The new value of the amplitude squared is

$$\varepsilon_{0,\text{new}}^2 = \frac{1}{\beta_L^2}(\Delta t_i)^2 + (\varepsilon_i - u_i)^2$$

$$= \frac{1}{\beta_L^2}(\Delta t_i)^2 + \varepsilon_i^2 - 2\varepsilon_i u_i + u_i^2$$

The change in the amplitude squared is

$$\delta\varepsilon_0^2 = \varepsilon_{0,new}^2 - \varepsilon_0^2 = -2\varepsilon_i u_i + u_i^2$$

If the rate of emission of photons is $\dot{N}$, then the instantaneous rate of change of the amplitude squared is

$$\frac{d\varepsilon_0^2}{dt} = -2\varepsilon_i \dot{N} u_i + \dot{N} u_i^2$$

$$= -2\varepsilon_i P_i + \dot{N} u_i^2$$

in which $P_i = \dot{N} u_i$ is the power radiated by the *i*th particle. The rate of change of the squared amplitude will vary at different points around the ring; we will be interested in the long term behavior, so we average over one turn:

$$\overline{\frac{d\varepsilon_0^2}{dt}} = \frac{1}{T_s}\oint_{turn} dt \frac{d\varepsilon_0^2}{dt} = -\frac{2}{T_s}\oint_{turn} dt \varepsilon_i P_i + \frac{1}{T_s}\oint_{turn} dt \dot{N} u_i^2$$

To find the rate of change of the amplitude squared for the whole beam, we average over all the particles, giving

$$\frac{d\varepsilon_0^2}{dt} = -\frac{2}{T_s}\oint_{turn}\langle dt\varepsilon_i P_i\rangle + \frac{1}{T_s}\oint_{turn} dt\langle \dot{N}u^2\rangle$$

in which we understand that ε_0^2 corresponds to the one-turn average, and T_s is the revolution period of the synchronous particle.

For a large number of particles, the average over the ensemble of particles $\langle \dot{N}u_i^2\rangle$ is the same as the average over the photon energy distribution, so $\langle \dot{N}u_i^2\rangle = \langle \dot{N}u^2\rangle$; we have made this replacement in the second term on the right. This term is always positive and represents the amplitude *growth* due to the fluctuations in the energy of the emitted photons. Since $\langle \varepsilon_i\rangle = 0$, the first term on the

right hand side would be zero, when we do the average, if P_i were independent of the energy. But it is not: its dependence on energy is precisely what causes *damping*. This energy dependence leads to terms like $\langle \varepsilon_i^2\rangle$, which are not zero and give damping.

To proceed, we need evaluate the explicit energy dependence of the integrand in $\oint_{turn}\langle dt\varepsilon_i P_i\rangle$ for a *non-synchronous particle*, and then do the average over all the particles.

The general path length differential is

$$dl = ds\left(1 + \frac{x}{\rho}\right)$$

and $c = \dfrac{dl}{dt}$

so

$$\oint_{turn}\langle dt\varepsilon_i P_i\rangle = \frac{1}{c}\oint_{turn}\langle dl\varepsilon_i P_i\rangle = \frac{1}{c}\oint_{turn}\left\langle ds\left(1+\frac{x}{\rho}\right)\varepsilon_i P_i\right\rangle$$

The energy dependence of *x*, for the *i*th particle, is given by

$$x_i = \eta\delta_i = \eta\frac{\varepsilon_i}{E_s}.$$

in which η is the dispersion function.

The energy dependence of the power results from the direct E^2 dependence, and also indirectly from the field dependence:

$$P_i(\varepsilon_i) = P_s\left(1 + 2\frac{1}{B_0}\frac{dB}{dE}\varepsilon_i + 2\frac{\varepsilon_i}{E_s}\right)$$

$$P_s = \frac{e^4 B_0^2\gamma_s^2}{6\pi\varepsilon_0 m_0^2 c} = \frac{e^2 c\gamma_s^4}{6\pi\varepsilon_0\rho^2}$$

The energy dependence resulting from the field arises in locations in which there is a field gradient $\frac{dB}{dx} = KB_0\rho$, and where there is also dispersion, so $x_i = \eta\delta_i = \eta\frac{\varepsilon_i}{E_s}$. Then

$$\frac{dB}{dE} = \frac{dB}{dx}\frac{dx}{dE} = K\frac{B_0\rho\eta}{E_s}$$

and we have

$$P_i(\varepsilon_i) = P_s\left(1 + \frac{2\varepsilon_i}{E_s}(K\rho\eta + 1)\right)$$

Putting this into the equation above for $\oint_{turn} \langle dt\varepsilon_i P_i\rangle$ gives

$$\oint_{turn}\left\langle ds\left(1+\frac{x}{\rho}\right)\varepsilon_i P_i\right\rangle = \oint_{turn}\left\langle dsP_s\left(1+\frac{\varepsilon_i}{E_s}\frac{\eta}{\rho}\right)\varepsilon_i\left(1+\frac{2\varepsilon_i}{E_s}(K\rho\eta+1)\right)\right\rangle$$

$$= \oint_{turn}\left\langle dsP_s\left(\varepsilon_i + \frac{\varepsilon_i^2}{E_s}\left[2+K\rho\eta+\frac{\eta}{\rho}\right] + \varepsilon_i^3\frac{2\eta(K\rho\eta+1)}{E_s^2\rho}\right)\right\rangle$$

Since $\varepsilon_i = \varepsilon_0\sin(2\pi Q_s n + \psi_i)$, then $\langle\varepsilon_i\rangle = \langle\varepsilon_i^3\rangle = 0$, and $\langle\varepsilon_i^2\rangle = \dfrac{\varepsilon_0^2}{2}$,

so the integral simplifies to

$$\oint_{turn}\langle dt\varepsilon_i P_i\rangle = \frac{\varepsilon_0^2}{2E_s}\oint_{turn} dsP_s\left[2+K\rho\eta+\frac{\eta}{\rho}\right]$$

Using the previously developed expressions for P_s and U_s gives

$$\oint_{turn}\langle dt\varepsilon_i P_i\rangle = \frac{\varepsilon_0^2 U_s}{2E_s}(2+\mathsf{D})$$

in which

$$\mathsf{D} = \frac{\int_0^C \frac{\eta}{\rho^2}ds\left(\frac{1}{\rho}+2K\rho\right)}{\int_0^C \frac{ds}{\rho^2}}$$

Returning to the equation for the time derivative of the amplitude squared, we have

$$\frac{d}{dt}\left(\varepsilon_0^2\right) = -\frac{\varepsilon_0^2 U_s}{T_s E_s}(2+\mathsf{D}) + \frac{1}{T_s}\oint_{turn} dt\langle\dot{N}u^2\rangle$$

in which the amplitude is understood to be averaged over one turn.

The first term on the right represents the amplitude reduction due to damping. The second term represents the amplitude growth due to fluctuations in photon energy.

We can integrate this equation to find the time dependence of the average amplitude squared:

$$\varepsilon_0(t)^2 = \varepsilon_0(0)^2\exp\left[-\frac{t}{\tau}\right] + \varepsilon_{0,\infty}^2\left(1-\exp\left[-\frac{t}{\tau}\right]\right)$$

$$\frac{1}{\tau} = \frac{U_s}{T_s E_s}(2+\mathsf{D}) \qquad \varepsilon_{0,\infty}^2 = \frac{\tau}{T_s}\oint_{turn} dt\langle\dot{N}u^2\rangle$$

The energy amplitude squared, ε_0^2, damps at the rate $1/\tau$;

the energy $\varepsilon_i \propto \sqrt{\varepsilon_0^2}$ damps at half the rate:

$$\frac{1}{\tau_\varepsilon} = \frac{1}{2\tau} = \frac{U_s}{2T_s E_s}(2+\mathsf{D})$$

For a *separated function isomagnetic* lattice, neglecting the dipole focusing terms, when K is non-zero, ρ is infinite, so

$$\mathrm{D} = \frac{\int_0^C \eta ds\left(\frac{1}{\rho^3} + 2\frac{K}{\rho}\right)}{\int_0^C \frac{ds}{\rho^2}} = \frac{\int_0^C \frac{\eta}{\rho^3} ds}{\int_0^C \frac{ds}{\rho^2}} = \frac{\frac{1}{\rho^2}\int_0^C \frac{\eta}{\rho} ds}{\frac{2\pi}{\rho}} = \frac{C\alpha_C}{2\pi\rho} << 1$$

In this case, we see that *the energy damping time τ_ε is just the time in which a particle would radiate away all its energy* . Specifically, we have

$$\frac{1}{\tau_\varepsilon} = \frac{r_0}{\rho}\frac{4\pi\gamma_s^3}{3T_s}$$

The damping rate grows like the energy cubed, and decreases like the square of the ring radius.

The final mean square value of the energy spread will be

$$\sigma_{\varepsilon,\infty}^2 = \left\langle \varepsilon_{i,\infty}^2 \right\rangle = \frac{\varepsilon_{0,\infty}^2}{2} = \frac{\tau_\varepsilon}{4T_s}\oint_{turn} dt \left\langle \dot{N}u^2 \right\rangle$$

$$= \frac{E_s}{2U_s(2+\mathrm{D})}\oint_{turn} dt \left\langle \dot{N}u^2 \right\rangle$$

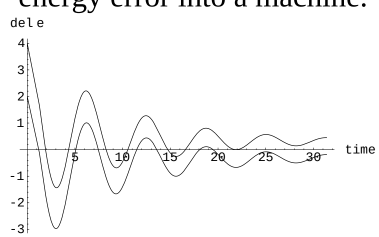
The final value is called the *equilibrium energy spread*.

The equilibrium energy spread is proportional to the root mean square fluctuations in the energy of the synchrotron radiation photons. Because the radiation is a statistical process, the final

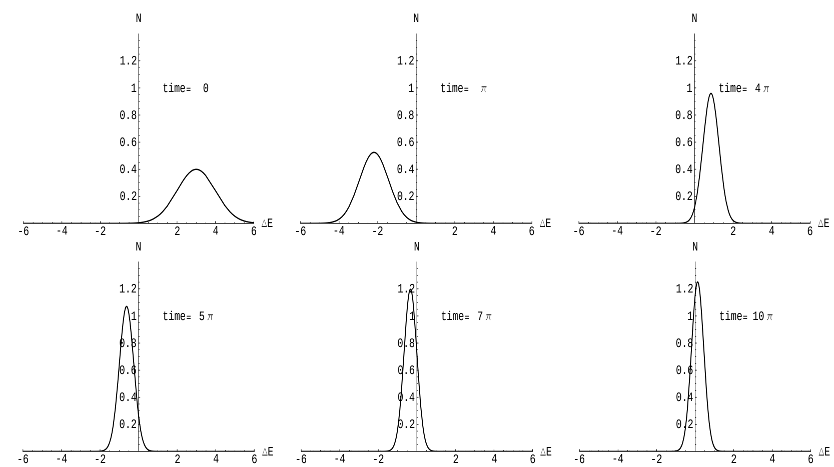
distribution in energy will be Gaussian, with $\sigma_{\varepsilon,\infty}^2$ as its mean square deviation from E_s

Example: damping of an injected electron beam.

The following figure shows the overall energy envelope of the beam, as a function of time, for an electron beam with an energy spread larger than the equilibrium energy spread, injected with an energy error into a machine.



The same process is shown in the next figures: in this case, the complete energy distribution is plotted, at various times:



Evaluation of the equilibrium energy spread:

Using $\dot{N} = \frac{15\sqrt{3}}{8}\frac{P}{u_c}, \quad \langle u^2 \rangle = \frac{11}{27}u_c^2, \quad u_c = \frac{3}{2}\frac{\hbar\gamma^3 c}{\rho}$, we get

$$\oint_{turn} dt \langle \dot{N}u^2 \rangle = \frac{55}{16\sqrt{3}} \frac{e^2\hbar c\gamma^7}{6\pi\varepsilon_0} \int_0^C ds \frac{1}{\rho^3}$$

and so

$$\sigma_{\varepsilon,\infty}^2 = \frac{E_s}{2U_s(2+D)} \oint_{turn} dt \langle \dot{N}u^2 \rangle = C_q \frac{\gamma_s^2 E_s^2}{(2+D)} \frac{\int_0^C ds \frac{1}{\rho^3}}{\int_0^C \frac{ds}{\rho^2}}$$

in which $C_q = \frac{55}{32\sqrt{3}}\frac{\hbar}{m_0 c} = 3.84 \times 10^{-13}$ m (for electrons).

For an isomagnetic lattice, this simplifies to

$$\sigma_{\varepsilon,\infty}^2 = C_q \frac{\gamma_s^2 E_s^2}{(2+D)\rho}$$

The mean square relative energy spread is then

$$\left(\frac{\sigma_{\varepsilon,\infty}}{E_s}\right)^2 = \frac{C_q}{\rho}\frac{\gamma_s^2}{(2+D)}$$

Note that, for a separated function isomagnetic lattice for which D <<1, the relative energy spread depends only on the energy and the ring radius. It grows linearly with the energy.

The corresponding mean square bunch length is

$$\sigma_{s\infty}^2 = c^2\sigma_{t\infty}^2 = c^2\sigma_{\varepsilon,\infty}^2\beta_L^2 = c^2\beta_L^2 C_q \frac{\gamma_s^2 E_s^2}{(2+D)\rho}$$

This can also be written as

$$\sigma_{s\infty}^2 = \lambda^2 \frac{C_q}{\rho}\frac{\gamma_s^2}{(2+D)}\frac{h\eta_C}{2\pi}\frac{-E_s}{eV\cos\phi_s} = \frac{C_q}{\rho}\frac{1}{(2+D)}\left(\frac{\lambda\gamma_s h\eta_C}{2\pi Q_s}\right)^2$$

The *equilibrium rms emittance* is

$$\varepsilon_{L,\infty} = \beta_L C_q \frac{\gamma_s^2 E_s^2}{(2+D)\rho}$$

In contrast to the energy spread, the bunch length (and the emittance) depend on many quantities: the slip factor, the rf voltage and wavelength, the synchrotron tune, and the harmonic number

Example:

CESR operates as a storage ring at E=5.29 GeV. We'll use the following approximations:

Take the lattice to be a separated function isomagnetic lattice with ρ=98 m. We are ignoring the "hard bend" regions, in which there is considerable additional synchrotron radiation; hence the enrgy spread and bunch length will be underestimated with this approximation.

The rf frequency is 500 MHz, and the rf voltage is 6.5 MV. α_C = 0.01077, *h*=1281.

Then we find the table below. The "true" column gives the numbers including the hard and soft bend regions.

Parameter	Isomag. model	True	Units
U_s	710	1140	keV
τ_ε	19	12	ms
ϕ_s	174	168	degrees
ε_L	93	200	eV-μs
Q_s	.052	.0514	
$1/Q_s$	19.3	19.5	
β_L	16	16	ps/MeV
σ_E	2.4	3.5	MeV
σ_E/E	456	673	$x10^{-6}$
σ_t	38	57	ps
σ_s	11.6	17	mm
ΔE_b	36.4	32.8	MeV
A_b/π	28982		eV-μs