LECTURE 13

Transition Crossing in Proton synchrotrons

Synchrotron radiation: transverse effects

Vertical damping

Horizontal damping and quantum excitation

Equilibrium horizontal emittance

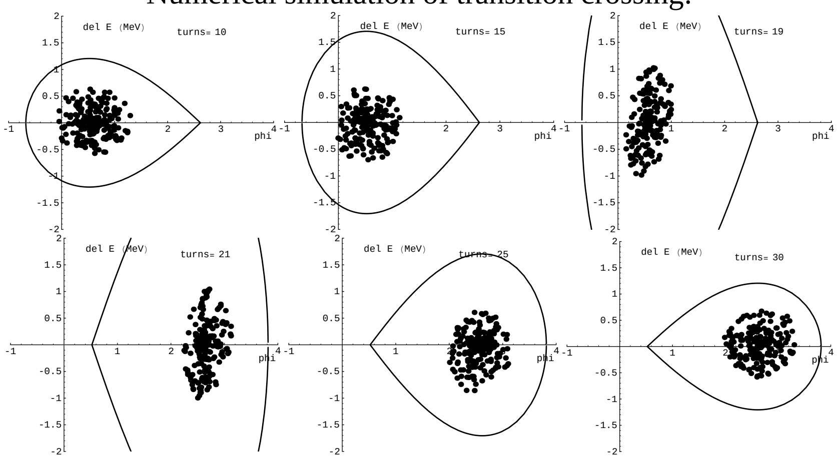
Transition Crossing in Proton synchrotrons

When γ passes through γ_t, the slip factor

$$\eta_c = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}$$

goes to zero. The synchrotron tune $Q_s^2 = -\frac{eVh\eta_C \cos\phi_s}{2\pi E_s \beta_s^2}$ likewise goes to zero, as does the longitudinal beta function $\beta_L = \frac{\lambda}{c\beta_s}\sqrt{-\frac{\eta_C h}{2\pi e V E_s \cos\phi_s}}$. Since $\frac{\Delta t}{\Delta E} = \beta_L$, this corresponds to a growth in the energy spread, and a reduction in the time spread, of the bunch.

Numerical simulation of transition crossing:



Possible problems which can occur at transition:

- Beam loss at high dispersion points due to the increased energy spread
- Transverse and longitudinal emittance growth due to chromatic nonlinear effects
- Increased susceptibility to various forms of beam instabilities due to loss of phase focusing
- Maintenance of beam loading compensation during the rf phase jump

Synchrotron radiation: transverse effects

In addition to the damping and quantum excitation effects in longitudinal phase space, synchrotron radiation also has important consequences for transverse beam dynamics. These effects are

1. A damping of both horizontal and vertical betatron oscillations.

2. Quantum excitation of horizontal betatron oscillations, leading to an equilibrium rms horizontal emittance.

In the absence of vertical dispersion, there are no quantum excitation effects in the vertical plane. Consequently, the vertical beam size typically damps to a much smaller value than the equilibrium horizontal beam size. Beams in electron machines are

thus typically “flat”, with a very different aspect ratio in the vertical and horizontal directions. The limit to the vertical beam size is typically set by effects like small coupling between the planes, and residual vertical dispersion. The vertical beam dimension is typically 10 percent of the horizontal dimension.

We will find *two important results*, which can be stated very simply:

1. The sum of the damping rates in all three planes (horizontal, vertical, and longitudinal) is a constant:

$$\frac{1}{\tau_\varepsilon} + \frac{1}{\tau_x} + \frac{1}{\tau_y} = \frac{2U_s}{E_s T_s}$$

This is called *Robinson’s theorem.* Although we will verify this by calculating $\frac{1}{\tau_x}$ and $\frac{1}{\tau_y}$ individually, and combining with $\frac{1}{\tau_\varepsilon}$ from the previous lecture, in fact Robinson’s theorem can be proved for a very general case, including arbitrary *x*-*y* coupling and vertical dispersion, for arbitrary lattice configurations.

For a *separated function* lattice, $\tau_x \approx \tau_y$ and $\tau_\varepsilon \approx T_s \frac{E_s}{U_s}$. In this case, $\tau_x \approx \tau_y \approx 2\tau_\varepsilon$

2. The equilibrium rms horizontal beam size is approximately the square root of the number of photons emitted during one damping time, times the average orbit offset (dispersion times energy offset divided by energy) which would be caused by an energy offset equal to the rms photon energy.

$$\sigma_x \approx \sqrt{\dot{N}\tau_x}\langle\eta\rangle \frac{\sqrt{\langle u^2 \rangle}}{E_s}$$

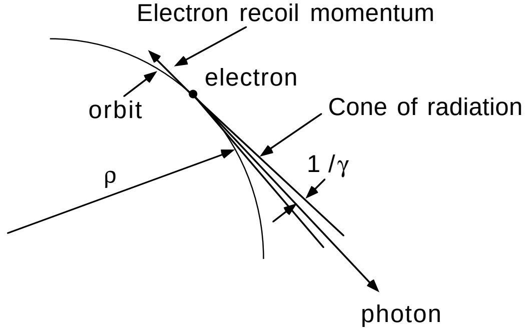
Example: CESR

The beam size damping time is about twice the energy damping time. The dispersion is about 1.5 m. The equilibrium beam size is then roughly

$$\sigma_x \approx \sqrt{\dot{N}\tau_x}\langle\eta\rangle \frac{\sqrt{\langle u^2 \rangle}}{E_s} \approx \sqrt{2 \times 7300 \times 721} \times 1.5 \text{ m} \times \frac{2.3 \text{ keV}}{5.2 \text{ GeV}}$$

≈ 2 mm

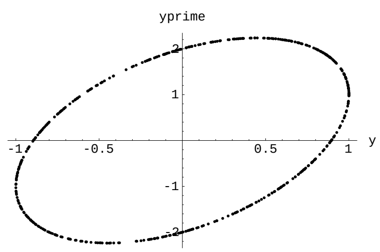
Let’s now see how to get more accurate estimates for these quantities. Before we begin, we have to understand an important feature of synchrotron radiation, which we have not considered before. This is the fact that the photons that are emitted by an accelerated charged particle are emitted in a cone of half-angle $1/\gamma$, around the direction of motion of the particle.



For relativistic particles, $1/\gamma$ is very small. Thus the direction of the recoil momentum change of the electron is almost exactly opposite to the electron's momentum: $\delta\vec{p} \propto -\hat{p}$

Vertical Damping

We'll start with a description of the vertical synchrotron radiation damping. Suppose that we have a collection of particles in vertical phase space, all having the same value of the vertical emittance, but distributed randomly around the ellipse:



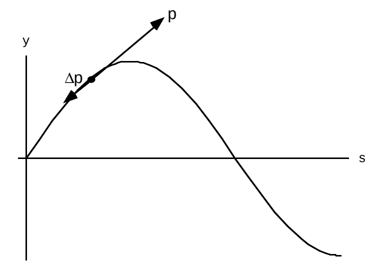
The solution of the trajectory equations for the *i*th particle is

$$y_i = a\sqrt{\beta}\cos\left(\Phi(s)+\psi_i\right)$$

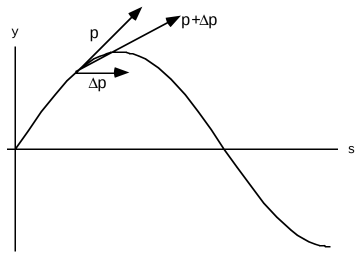
$$y_i' = -\frac{a}{\sqrt{\beta}}\left(\alpha\cos\left(\Phi(s)+\psi_i\right)+\sin\left(\Phi(s)+\psi_i\right)\right)$$

Now let each particle radiate a photon of energy u_i. The energy of the particle changes according to $\varepsilon_i \to \varepsilon_i - u_i$. What happens to the position and slope of the trajectory? The answer: *nothing*, to a very good approximation. Why? The change in energy clearly does not

change the position of the particle. Since the direction of the recoil momentum change is opposite to that of the momentum, the slope of the trajectory also does not change.



However: the energy which is lost must be restored by the rf, on every turn. The electric field in the rf cavity exerts a force on the particle in the *s* direction, causing a momentum change in that direction. If the particle is undergoing a betatron oscillation, this momentum change is *not* parallel to the particle's momentum and does result in a change in the slope.



The change in the slope is given by

$$y' = \frac{p_y}{p}; y' + \Delta y' = \frac{p_y}{p + \Delta p} \approx \frac{p_y}{p}\left(1 - \frac{\Delta p}{p}\right)$$

$$\Delta y' = -y' \frac{\Delta p}{p} \approx -y' \frac{\Delta E}{E}$$

For the ith particle, the energy change (the energy restored by the rf) is equal to the sum of the energies emitted by all the photons emitted on one turn, which is the total energy loss per turn, U_s . Thus, the change in slope at the rf is

$$\Delta y_i' = -y_i' \frac{U_s}{E_s}$$

The amplitude of the y-oscillation is given by

$$a^2 = \gamma y_i^2 + 2\alpha y_i y_i' + \beta y_i'^2$$

The change in this amplitude is

$$\Delta a^2 = 2\alpha y_i \Delta y_i' + 2\beta y_i' \Delta y' = -\frac{2U_s}{E_s}\left(\alpha y_i y_i' + \beta y_i'^2\right)$$

$$= -\frac{2U_s}{E_s}\begin{pmatrix} -\alpha a^2 \cos(\Phi(s)+\psi_i)\left(\alpha\cos(\Phi(s)+\psi_i) + \sin(\Phi(s)+\psi_i)\right) \\ +a^2\left(\alpha\cos(\Phi(s)+\psi_i) + \sin(\Phi(s)+\psi_i)\right)^2 \end{pmatrix}$$

To get the amplitude change for the whole beam, average over the random phases ψ_i. Averages of cosine squared and sine squared become 1/2, while averages of products of sine and cosine are zero. After doing the average, we get

$$\Delta a^2 = -\frac{U_s}{E_s}\left(-\alpha^2 a^2 + a^2\left(1+\alpha^2\right)\right) = -a^2 \frac{U_s}{E_s}$$

This is the change in the amplitude for one turn: so

$$\frac{da^2}{dt} = -a^2 \frac{U_s}{T_s E_s}$$

Since $\langle y^2 \rangle = \sigma_y^2 = \frac{a^2}{2}$, we have

$$\frac{d\sigma_y^2}{dt} = 2\sigma_y \frac{d\sigma_y}{dt} = -\sigma_y^2 \frac{U_s}{T_s E_s} \Rightarrow$$

$$\frac{d\sigma_y}{dt} = -\sigma_y \frac{U_s}{2T_s E_s} = -\frac{\sigma_y}{\tau_y}$$

So we see the damping time in y is

$$\tau_y = 2T_s \frac{E_s}{U_s}$$

<u>Horizontal Damping</u>

Horizontal synchrotron radiation damping is a little more complicated. Suppose that we have a collection of particles in horizontal phase space, all having the same value of the horizontal emittance, but distributed randomly around the ellipse.

The solution of the trajectory equations for the *i*th particle is

$$x_i = x_{\beta,i} + \eta \frac{\varepsilon_i}{E_s}$$

$$x_i' = x_{\beta,i}' + \eta' \frac{\varepsilon_i}{E_s}$$

in which the betatron motion is given by

$$x_{\beta,i} = a\sqrt{\beta}\cos\left(\Phi(s) + \psi_i\right)$$

$$x_{\beta,i}' = -\frac{a}{\sqrt{\beta}}\left(\alpha\cos\left(\Phi(s)+\psi_i\right) + \sin\left(\Phi(s)+\psi_i\right)\right)$$

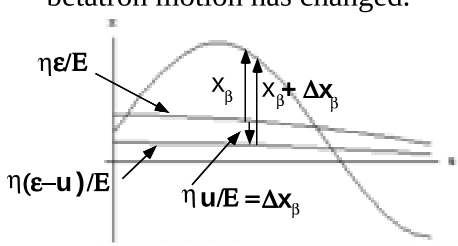
. Now let each particle radiate a photon of energy u_i. The energy of the particle changes according to $\varepsilon_i \to \varepsilon_i - u_i$. What happens to the position and slope of the trajectory? The answer is still “nothing”, but in this case, because of the *dispersion* at the site of the photon emission, the betatron motion will change according to

$$\Delta x_i = 0 = \Delta x_{\beta,i} - \eta \frac{u_i}{E_s}$$

$$\Delta x_i' = 0 = \Delta x_{\beta,i}' - \eta' \frac{u_i}{E_s}$$

Essentially, what has happened is that, since the energy has changed, the orbit with respect to which betatron motion is measured has changed. But since the particle itself is still in the

same place, it is now a different distance from that orbit, so its betatron motion has changed.



The amplitude of the *x* betatron oscillation is given by

$$a^2 = \gamma x_{\beta,i}^2 + 2\alpha x_{\beta,i}x_{\beta,i}' + \beta x_{\beta,i}'^2$$

The change in this amplitude is

$$\Delta a^2 = \gamma\left(x_{\beta,i} + \Delta x_{\beta,i}\right)^2 + 2\alpha\left(x_{\beta,i} + \Delta x_{\beta,i}\right)\left(x_{\beta,i}' + \Delta x_{\beta,i}'\right)$$

$$+\beta\left(x_{\beta,i}' + \Delta x_{\beta,i}'\right)^2 - \left(\gamma x_{\beta,i}^2 + 2\alpha x_{\beta,i}x_{\beta,i}' + \beta x_{\beta,i}'^2\right)$$

$$= \frac{2u_i}{E_s}\left(\alpha\left(\eta' x_{\beta,i} + \eta x_{\beta,i}'\right) + \eta\gamma x_{\beta,i} + \beta\eta' x_{\beta,i}'\right) + \left(\frac{u_i}{E_s}\right)^2 \mathcal{H}$$

in which

$$\mathcal{H} = \eta^2\gamma + 2\alpha\eta\eta' + \beta\eta'^2$$

The instantaneous rate of change of the betatron amplitude squared is obtained by multiplying by the photon emission rate $\dot{N}$:

$$\frac{da^2}{dt} = \frac{2P_i}{E_s}\left(\alpha\left(\eta' x_{\beta,i} + \eta x_{\beta,i}'\right) + \eta\gamma x_{\beta,i} + \beta\eta' x_{\beta,i}'\right) + \dot{N}\left(\frac{u_i}{E_s}\right)^2 \mathcal{H}$$

in which we've used $P_i = \dot{N}u_i$. This quantity will vary around the ring; we're interested in the average over one turn:

$$\overline{\frac{da^2}{dt}} = \frac{1}{T_s}\oint_{turn} dt\,\frac{da^2}{dt}$$

$$= \frac{2}{T_s E_s}\oint_{turn} dt P_i\left(\alpha\left(\eta' x_{\beta,i} + \eta x'_{\beta,i}\right) + \eta\gamma x_{\beta,i} + \beta\eta' x'_{\beta,i}\right)$$

$$+ \frac{1}{T_s E_s^2}\oint_{turn} dt \dot{N}u_i^2\, \mathsf{H}$$

To get the amplitude change for the whole beam, average over all the particles.

$$\frac{da^2}{dt} = \frac{2}{T_s E_s}\oint_{turn}\left\langle dt P_i\left(\alpha\left(\eta' x_{\beta,i} + \eta x'_{\beta,i}\right) + \eta\gamma x_{\beta,i} + \beta\eta' x'_{\beta,i}\right)\right\rangle$$

$$+ \frac{1}{T_s E_s^2}\oint_{turn} dt\left\langle \dot{N}u^2\right\rangle \mathsf{H}$$

in which we understand that a^2 corresponds to the one-turn average of the amplitude squared.

For a large number of particles, the average over the ensemble of particles $\left\langle \dot{N}u_i^2\right\rangle$ is the same as averages over the photon energy distribution, so $\left\langle \dot{N}u_i^2\right\rangle = \left\langle \dot{N}u^2\right\rangle$, so we have made this replacement in the second term on the right. To evaluate the first term on the right, we must write out the explicit dependence on $x_{\beta,i}$ and $x'_{\beta,i}$, substitute in for these quantities from the trajectory solutions, and

average over the random phases of the particles. Using

$$dl = ds\left(1 + \frac{x}{\rho}\right) \text{ and } c = \frac{dl}{dt}, \text{ we have}$$

$$\oint_{turn}\left\langle dt P_i\left(\alpha\left(\eta' x_{\beta,i} + \eta x'_{\beta,i}\right) + \eta\gamma x_{\beta,i} + \beta\eta' x'_{\beta,i}\right)\right\rangle$$

$$= \frac{1}{c}\int_0^C ds\left\langle\left(1 + \frac{x_{\beta,i}}{\rho}\right) P_i(x_{\beta,i})\left(\alpha\left(\eta' x_{\beta,i} + \eta x'_{\beta,i}\right) + \eta\gamma x_{\beta,i} + \beta\eta' x'_{\beta,i}\right)\right\rangle$$

The dependence of the power on $x_{\beta,i}$ comes through the field:

$$P_i(x_{\beta,i}) = P_s\left(1 + 2\frac{1}{B_0}\frac{dB}{dx}x_{\beta,i}\right) = P_s\left(1 + 2K\rho x_{\beta,i}\right)$$

Plug this in; then use

$$x_{\beta,i} = a\sqrt{\beta}\cos(\Phi(s) + \psi_i)$$

$$x'_{\beta,i} = -\frac{a}{\sqrt{\beta}}\left(\alpha\cos(\Phi(s) + \psi_i) + \sin(\Phi(s) + \psi_i)\right)$$

Averages of cosine squared and sine squared become 1/2, while averages of products of sine and cosine are zero. After doing the average, we get

$$\oint_{turn}\left\langle dt P_i\left(\alpha\left(\eta' x_{\beta,i} + \eta x'_{\beta,i}\right) + \eta\gamma x_{\beta,i} + \beta\eta' x'_{\beta,i}\right)\right\rangle$$

$$= \frac{a^2}{2c}\int_0^C ds P_s\frac{\eta}{\rho}\left(1 + 2K\rho^2\right) = \frac{a^2 U_s}{2}\frac{\int_0^C ds\frac{\eta}{\rho^3}\left(1 + 2K\rho^2\right)}{\int_0^C \frac{ds}{\rho^2}} = \frac{a^2 U_s}{2}\mathsf{D}$$

The equation for a^2 then becomes

$$\frac{da^2}{dt} = \frac{a^2 U_s}{T_s E_s}\mathcal{D} + \frac{1}{T_s E_s^2}\oint_{turn} dt \left\langle \dot{N}u^2 \right\rangle \mathcal{H}$$

We now have to include the effect of the energy restoration by the rf cavities. The argument here is identical to that given for the vertical damping: we get an extra contribution equal to

$$\frac{da^2}{dt} = -a^2 \frac{U_s}{T_s E_s}$$

$$\frac{da^2}{dt} = -\frac{a^2 U_s}{T_s E_s}(1-\mathcal{D}) + \frac{1}{T_s E_s^2}\oint_{turn} dt \left\langle \dot{N}u^2 \right\rangle \mathcal{H}$$

The first term on the right represents the amplitude reduction due to damping. Note that, for $\mathcal{D}$<<1, as in a separated function lattice, essentially all the damping comes from the rf cavity energy

restoration. The second term represents the amplitude growth due to fluctuations in photon energy in dispersive regions.

We can integrate this equation to find the time dependence of the average amplitude squared:

$$a(t)^2 = a(0)^2 \exp\left[-\frac{t}{\tau}\right] + a_\infty^2\left(1 - \exp\left[-\frac{t}{\tau}\right]\right)$$

$$\frac{1}{\tau} = \frac{U_s}{T_s E_s}(1-\mathcal{D}) \qquad a_\infty^2 = \frac{\tau}{E_s^2 T_s}\oint_{turn} dt \left\langle \dot{N}u^2 \right\rangle \mathcal{H}$$

The position $x_i \propto \sqrt{a^2}$ damps at half the rate:

$$\frac{1}{\tau_x} = \frac{1}{2\tau} = \frac{U_s}{2T_s E_s}(1-\mathcal{D})$$

We see explicitly that the Robinson Theorem is verified:

$$\frac{1}{\tau_\varepsilon} + \frac{1}{\tau_x} + \frac{1}{\tau_y} = \frac{U_s}{2T_s E_s}(2+\mathcal{D}) + \frac{U_s}{2T_s E_s}(1-\mathcal{D}) + \frac{U_s}{2T_s E_s} = \frac{2U_s}{E_s T_s}$$

and that, for small $\mathcal{D}$, $\tau_x = \tau_y = 2\tau_\varepsilon$.

The final mean square value of the position spread will be

$$\sigma_{x,\infty}^2 = \left\langle x_{i,\infty}^2 \right\rangle = \frac{a_\infty^2 \beta}{2} = \frac{\tau_x \beta}{4E_s^2 T_s}\oint_{turn} dt \left\langle \dot{N}u^2 \right\rangle \mathcal{H}$$

$$= \frac{\beta}{2U_s E_s (1-\mathcal{D})}\oint_{turn} dt \left\langle \dot{N}u^2 \right\rangle \mathcal{H}$$

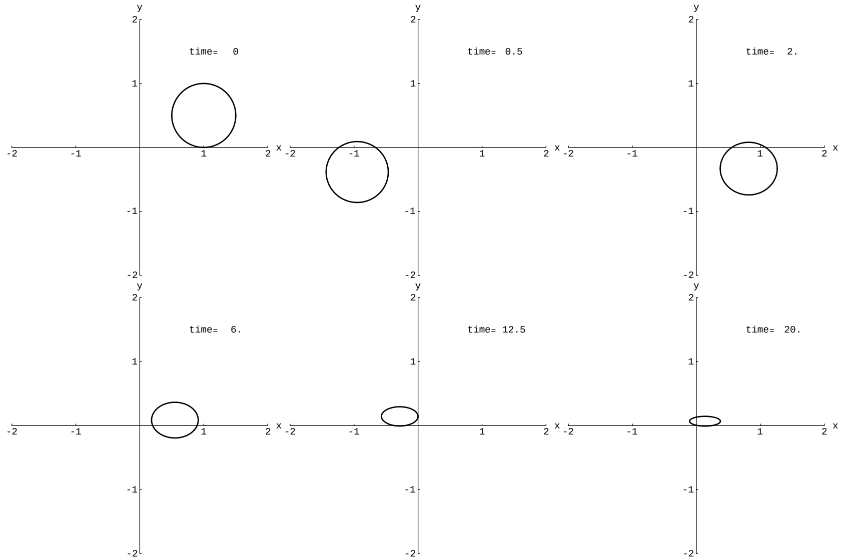
The final value is called the *equilibrium rms beam size.*

The equilibrium beam size is proportional to the mean square fluctuations in the energy of the synchrotron radiation photons. Because the radiation is a statistical process, the final distribution in position will be Gaussian, with $\sigma_{x,\infty}^2$ as its mean square.

Example: damping of an injected electron beam.

This proceeds just as we described the energy damping of an injected beam with an energy offset. If an electron beam is injected off-axis, with a coherent betatron oscillation, the centroid oscillation will damp to the reference orbit with the damping rate given above, while the beam size damps to the equilibrium beam size at the same rate.

In the following figures, a round beam damps to a flat one. The damping time is τ=10 time units.



Equilibrium horizontal emittance

The *equilibrium rms horizontal emittance* is

$$\varepsilon_{x,\infty} = \frac{\sigma_{x,\infty}^2}{\beta} = \frac{\oint_{turn} dt\left\langle \dot{N}u^2 \right\rangle \mathcal{H}}{2U_s E_s(1-\mathcal{D})}$$

Evaluation of the equilibrium emittance:

Using $\dot{N} = \frac{15\sqrt{3}}{8}\frac{P}{u_c}$, $\left\langle u^2 \right\rangle = \frac{11}{27}u_c^2$, $u_c = \frac{3}{2}\frac{\hbar\gamma^3 c}{\rho}$, we get

$$\oint_{turn} dt\left\langle \dot{N}u^2 \right\rangle \mathcal{H} = \frac{55}{16\sqrt{3}}\frac{e^2\hbar c\gamma^7}{6\pi\varepsilon_0}\int_0^C ds\frac{\mathcal{H}}{\rho^3}$$

and so

$$\varepsilon_{x,\infty} = \frac{1}{2U_s E_s(1-\mathcal{D})}\oint_{turn} dt\left\langle \dot{N}u^2 \right\rangle \mathcal{H} = C_q\frac{\gamma_s^2}{(1-\mathcal{D})}\frac{\int_0^C ds\frac{\mathcal{H}}{\rho^3}}{\int_0^C \frac{ds}{\rho^2}}$$

in which $C_q = \frac{55}{32\sqrt{3}}\frac{\hbar}{m_0 c} = 3.84\times 10^{-13}$ m (for electrons).

For an isomagnetic lattice, this simplifies to

$$\varepsilon_{x,\infty} = C_q\frac{\gamma_s^2}{2\pi\rho(1-\mathcal{D})}\int_0^C ds\frac{\mathcal{H}}{\rho}$$

Note that the equilibrium emittance increases with the square of the energy, and decreases with the size of the ring. A crude estimate of $\varepsilon_{x,\infty}$ can be obtained for a separated function machine by using the approximation

$$\int_0^C ds\frac{\mathcal{H}}{\rho} \approx \left\langle \mathcal{H} \right\rangle \int_0^C \frac{ds}{\rho} \approx 2\pi\left\langle \frac{\eta^2}{\beta} \right\rangle$$

then, using $\langle\eta\rangle \approx \alpha_C R$, $\langle\beta\rangle \approx \frac{R}{Q_x}$, where $R = \frac{C}{2\pi}$ is the mean radius of the machine, we have

$$\varepsilon_{x,\infty} \approx \gamma_s^2\frac{C_q}{\rho}\left\langle \frac{\eta^2}{\beta} \right\rangle \approx C_q\gamma_s^2\frac{R}{\rho}\alpha_C^2 Q_x \approx C_q\gamma_s^2\frac{R}{\rho}\frac{1}{Q_x^3}$$

Because of the absence of quantum excitations in the vertical plane, the vertical equilibrium rms emittance will typically be much smaller than the horizontal emittance. It is usually determined by the typically small coupling between horizontal and vertical motion that exists in any real accelerator. Usually, the coupling is described in terms of a coupling coefficient κ, defined such that the sum of the horizontal and vertical emittances is equal to $\varepsilon_{x,\infty}$:

$$\varepsilon_x = \varepsilon_{x,\infty}\frac{1}{1+\kappa} \qquad \varepsilon_y = \varepsilon_{x,\infty}\frac{\kappa}{1+\kappa}$$

In an uncorrected machine, κ can be as large as 10%; with effort, it can be reduced to 1% or so.

With κ=1%, the vertical beam size will be about 10% of the horizontal beam size: the beam will have a 10 to 1 horizontal to vertical aspect ratio, and will be "flat".

Deliberate introduction of coupling can give κ=1; this is a "round beam".

Control of κ is crucial to the performance of colliding beam machines operating with "flat" beams, since the vertical beam size at the collision point directly determines the luminosity.

Example: transverse beam sizes in CESR

We'll use the following approximations:

Take the lattice to be a separated function isomagnetic lattice with ρ=98 m. Using the results above, and the crude approximation for the equilibrium emittance (with α_C=0.01077, Q_x=9.55), we get the following table. The "true" column gives the correct numbers including the hard and soft bend regions. The average beam sizes are calculated from

$$\sigma = \sqrt{\sigma_\beta^2 + \eta^2\left(\frac{\sigma_\varepsilon}{E_s}\right)^2} \approx \sqrt{\varepsilon\langle\beta\rangle + \langle\eta\rangle^2\left(\frac{\sigma_\varepsilon}{E_s}\right)^2}$$

$$\text{with } \langle\eta\rangle \approx \alpha_C R, \;\; \langle\beta\rangle \approx \frac{R}{Q_x}.$$

The numbers in the "true" column take into consideration the non-isomagnetic "hard" and "soft" bends near the interaction point in CESR. The coupling is taken as κ=0.016.

Parameter	Isomag. model	True	Units
τ_x	38	24	ms
τ_y	38	24	ms
ε_x	55	201	nm-rad
ε_y	0.9	3.1	nm-rad
σ_E/E	456	673	$\times 10^{-6}$
σ_x(avg)	1.0	1.83	mm
σ_y(avg)	0.107	0.2	mm