LECTURE 14

Non-linear transverse motion

Floquet transformation

Harmonic analysis-one dimensional resonances

Two-dimensional resonances

Non-linear transverse motion

Non-linear field terms in the trajectory equation:

Trajectory equation from Lecture 3, p 7, keeping only lowest order terms in the field errors ΔB:

$$z'' + K(s)z = -\frac{\Delta B(x,y,s)}{B_0\rho}$$

in which $z = x$ or y.

Nonlinear driving terms on the right-hand side can drive resonances in the transverse plane, leading to chaotic and ultimately unstable motion.

Non-linear terms can arise in the trajectory equations from a variety of sources:

- Sextupoles introduced to control the chromaticity.
- Errors in dipole and quadrupole magnets
- Higher multipole fields (e.g., octupoles), that, like the sextupoles, are introduced into the machine to control certain machine parameters.

- Coherent fields produced by the beam itself, such as space charge
- The beam-beam interaction, which, for a colliding beam machine, is usually the dominant source of nonlinear fields

Nonlinear fields are often deliberately introduced in order to manipulate the beam in transverse phase space: the most common example of this is resonant extraction, a technique used to extract the beam slowly from an accelerator.

The sensitivity of the beam to a nonlinear resonance depends on the magnitude and azimuthal distribution of the nonlinear fields that drive the resonance, the emittance of the beam, and the exact value of the fractional part of the tune.

In order to understand this quantitatively, we will solve the differential equation of motion with the nonlinear terms, using a perturbation method. To simplify the solution, we first make a change of variables.

Floquet transformation

The general trajectory equation of motion is

$$\frac{d^2 z}{ds^2} + K(s) z = -\frac{\Delta B(x,y,s)}{B_0 \rho}$$

in which z stands for x or y, and ΔB is a general nonlinear field.

We want to make a change of variables: from (z,s) to (ξ,ψ), where,

$$\xi = \frac{z}{\sqrt{\beta}} \quad \text{and} \quad \psi = \frac{\Phi(s)}{Q} = \frac{1}{Q}\int \frac{ds}{\beta}$$

Interpretation of the Floquet coordinates (ξ,ψ):

For ΔB=0, the solution to the trajectory equations is

$$z = a\sqrt{\beta}\cos(\Phi(s)+\phi)$$

$$z' = -\frac{a}{\sqrt{\beta}}\left(\alpha\cos(\Phi(s)+\phi)+\sin(\Phi(s)+\phi)\right)$$

and the Courant-Snyder invariant is

$$a^2 = \gamma z^2 + 2\alpha z z' + \beta z'^2$$

which corresponds to an ellipse in (z,z') phase space, with a shape and orientation which is a function of s.

In terms of Floquet coordinates:

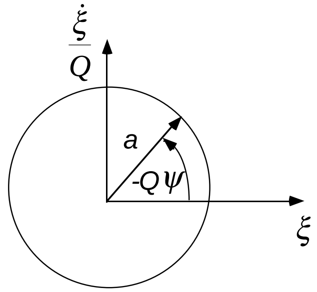
$$\xi(\psi) = \frac{z}{\sqrt{\beta}} \quad = a\cos(Q\psi+\phi)$$

$$\dot{\xi} = \frac{d\xi}{d\psi} = -Qa\sin(Q\psi+\phi)$$

The invariant is

$$a^2 = \xi^2 + \left(\frac{\dot{\xi}}{Q}\right)^2$$

which corresponds to a **circle** in $(\xi, \frac{\dot{\xi}}{Q})$ phase space, for all s.



In one turn, ψ advances by 2π, and the particle travels around the circle Q times per revolution.

The cosinelike and sinelike trajectories are very simple in these coordinates:

$$C(\psi,\psi_0) = \cos Q(\psi-\psi_0) \quad S(\psi,\psi_0) = \frac{\sin Q(\psi-\psi_0)}{Q}$$

$$C'(\psi,\psi_0) = -Q\sin Q(\psi-\psi_0) \quad S'(\psi,\psi_0) = \cos Q(\psi-\psi_0)$$

The one-turn matrix for these coordinates is the same everywhere in the machine, and is given by

$$\mathbf{M} = \begin{pmatrix} \cos 2\pi Q & \frac{\sin 2\pi Q}{Q} \\ -Q\sin 2\pi Q & \cos 2\pi Q \end{pmatrix}$$

If I know the coordinates $\xi, \dot{\xi}$, then the real space coordinates can be obtained from

$$z = \xi\sqrt{\beta}$$

$$z' = \frac{d}{ds}\left(\xi\sqrt{\beta}\right) = \sqrt{\beta}\frac{d\xi}{ds} - \xi\frac{\alpha}{\sqrt{\beta}}$$

$$= \sqrt{\beta}\frac{d\xi}{d\psi}\frac{d\psi}{ds} - \xi\frac{\alpha}{\sqrt{\beta}} = \frac{1}{Q\sqrt{\beta}}\left(\dot{\xi} - \alpha Q\xi\right)$$

The purpose of introducing these coordinates is that the linear, unperturbed motion is very simple in these coordinates.

When we introduce perturbations, they will deform the motion from that of a circle. To see how this happens, we must rewrite the general trajectory equation of motion in terms of the Floquet coordinates.

Differentiate z' given above and simplify:

$$z'' = \frac{\ddot{\xi} - Q^2\xi\left(\alpha^2 + \beta\alpha'\right)}{Q^2\beta^{3/2}}$$

Then the trajectory equation is

$$z'' + Kz = \frac{\ddot{\xi} - Q^2\xi\left(\alpha^2 + \beta\alpha' - K\beta^2\right)}{Q^2\beta^{3/2}} = -\frac{\Delta B(x,y,s)}{B_0\rho}$$

Recall: Lecture 5, p 22: in the derivation of Hill's equation, we found a differential equation for $\sqrt{\beta}$:

$$\frac{1}{(\sqrt{\beta})^3} + \frac{K}{\sqrt{\beta}} + \left(\sqrt{\beta}\right)'' = 0$$

This can be expanded to

$$2\beta\beta'' - \beta'^2 + 4\left(K\beta^2 - 1\right) = 0 \Rightarrow$$

$$\beta\alpha' + \alpha^2 = K\beta^2 - 1$$

So the trajectory equation simplifies to

$$\ddot{\xi} + Q^2\xi = -Q^2\beta^{3/2}\frac{\Delta B}{B_0\rho}$$

as we would expect, since we know the solution, for ΔB=0, is

$$\xi(\psi) = a\cos(Q\psi + \phi)$$

This is the equation of a <u>driven oscillator</u>.

For a general driving force of the form $A\exp[i\nu\psi]$

$$\ddot{\xi} + Q^2\xi = A\exp[i\nu\psi]$$

the inhomogeneous solution to this equation has the form

$$\xi(\psi) = a\exp[i\nu\psi]$$

Substituting this in the driven oscillator equation, we have

$$-a\nu^2\exp[i\nu\psi] + Q^2 a\exp[i\nu\psi] = A\exp[i\nu\psi] \Rightarrow$$

$$a = \frac{A}{Q^2 - \nu^2}$$

If the frequency of the driving force is very close to the natural frequency Q of the oscillator, the amplitude a of the driven

oscillations can be very large: this is a **resonance.** The **resonance condition** in this case is $Q = \pm\nu$.

This is the basic idea behind non-linear resonances in accelerators. Since the structure of the driving term is a more complex than a single harmonic function, a bit more analysis is required to get the details right.

<u>Harmonic Analysis</u>

Return to the trajectory equation in Floquet coordinates:

$$\ddot{\xi} + Q^2\xi = -Q^2\beta^{3/2}\frac{\Delta B}{B_0\rho}$$

Let us consider x motion, and a general nonlinear field of the form

$$\Delta B(x,s) = b_n(s)x^n$$

where b_n represents some field derivative: e.g, for n=2 (a sextupole field), $b_2 = \frac{B''}{2}$. If we plug this in to the driving term in the trajectory equation, that term becomes

$$-Q^2 \beta^{3/2} \frac{b_n x^n}{B_0 \rho}$$

Unfortunately, we don't know x, since that's what we're solving for. To go further, we make the <u>approximation</u> that the driving term is a small correction (a perturbation) to the motion, so we can approximate $x = \xi\sqrt{\beta}$ by using the linear motion result $\xi(\psi) = a\cos Q\psi$. Then, we have for the driving term, written as a function of ψ,

$$-Q^2 \left[\beta(s(\psi))^{(3+n)/2} \frac{b_n(s(\psi))}{B_0 \rho} \right] a^n \cos^n Q\psi$$

The quantity in brackets is a periodic function of s with period C, which means it is a periodic function of ψ with period 2π. So, it can be expressed as a Fourier expansion

$$\beta(s(\psi))^{(3+n)/2} \frac{b_n(s(\psi))}{B_0 \rho} = \sum_{m=-\infty}^{\infty} C_{m,n} \exp[im\psi]$$

where

$$C_{m,n} = \frac{1}{2\pi} \int_0^{2\pi} d\psi' \beta(s(\psi'))^{(3+n)/2} \frac{b_n(s(\psi'))}{B_0 \rho} \exp[-im\psi']$$

$$= \frac{1}{2\pi} \int_0^{C} \frac{d\psi'}{ds'} ds' \beta(s(\psi'))^{(3+n)/2} \frac{b_n(s(\psi'))}{B_0 \rho} \exp[-im\psi']$$

$$= \frac{1}{2\pi Q} \int_0^{C} ds' \beta(s')^{(1+n)/2} \frac{b_n(s')}{B_0 \rho} \exp\left[-im \frac{\Phi(s')}{Q}\right]$$

is the mth azimuthal Fourier coefficient for the field error $b_n(s)$. The Fourier coefficients describe how the field error (weighted by the appropriate power of β and phase advance) is distributed around the ring.

Examples:

1. A single field error at location s_0, of length L. We can choose Φ to be zero at this point: then

$$C_{m,n} = \frac{1}{2\pi Q} \beta(s_0)^{(1+n)/2} \frac{b_n(s_0) L}{B_0 \rho}$$

is independent of m: all values of m are present in the Fourier spectrum.

2. A machine with *superperiodicity N*: The lattice functions and the field errors are periodic in s with period $\frac{C}{N}$, where C is the circumference. For example, a machine made entirely of N FODO cells has superperiodicity N. In this case,

$$C_{m,n} = \begin{cases} \dfrac{N}{2\pi Q} \displaystyle\int_0^{C/N} ds' \beta(s')^{(1+n)/2} \dfrac{b_n(s')}{B_0 \rho} \exp\left[-im \dfrac{\Phi(s')}{Q}\right] & \text{for } m = jN \\ 0 & \text{for } m \neq jN \end{cases}$$

where j is any integer.

The Fourier coefficients are non-zero only for $m = 0, \pm N, \pm 2N, \pm 3N, \ldots$

Actual machines typically have low values of the superperiodicity, e.g, 1 (no symmetry), 2 (half-ring symmetry), 6 (six-fold symmetry), etc. A high value of N is very desirable, because of the elimination of many of the resonance-driving Fourier coefficients.

The driving term can now be written as

$$-Q^2 \sum_{m=-\infty}^{\infty} C_{m,n} \exp[im\psi] a^n \cos^n Q\psi$$

We want to get this in the form of a series of exponentials. We use the identity

$$\cos^n Q\psi = \frac{1}{2^n} \sum_{\substack{k=-n \\ \Delta k = 2}}^{n} \binom{n}{\frac{n-k}{2}} \exp(ikQ\psi)$$

where $\binom{n}{m} = \dfrac{n!}{m!(n-m)!}$ is the binomial coefficient. Then the driving term has the form

$$-Q^2 \left(\frac{a}{2}\right)^n \sum_{\substack{k=-n \\ \Delta k = 2}}^{n} \binom{n}{\frac{n-k}{2}} \sum_{m=-\infty}^{\infty} C_{m,n} \exp[i\psi(m + Qk)]$$

This is a sum of terms, each of which has the form of an exponential driving term. For each term in the sum, there is a possible resonance condition, given by

$$Q = \pm(m + Qk) \Rightarrow$$

$$Q(1 \mp k) = \pm m, \; -\infty \le m \le \infty, -n \le k \le n, \Delta k = 2$$

For a particular resonance, associated with the pair of integers (m,k), the driving term's strength is proportional to

$$-\left(\frac{a}{2}\right)^n \binom{n}{\frac{n-k}{2}} \frac{Q}{2\pi} \int_0^C ds' \beta(s')^{(1+n)/2} \frac{b_n(s')}{B_0 \rho} \cos[\Phi(s) - (1-k)\Phi(s')]$$

The value of $|1-k|$ is called the **order** of the resonance.

We can make the following table, which covers resonances due to dipole, quadrupole, sextupole and octupole field errors:

Field error type	n	k	Order $\lvert 1-k \rvert$	Resonant values of the tune $Q_{res} = \frac{m}{1-k}$, $m = 0, 1, 2, \ldots$
dipole	0	0	1	m: $1, 2, 3, 4, \ldots$
quadrupole	1	1	0	tune shift: m=0
quadrupole	1	-1	2	$\frac{m}{2}$: $\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots$

sextupole	2	2	1	m: 1,2,3,4,...
sextupole	2	0	1	m: 1,2,3,4,...
sextupole	2	-2	3	$\frac{m}{3}$: $\frac{1}{3},\frac{2}{3},1,\frac{4}{3},...$
octupole	3	3	2	$\frac{m}{2}$: $\frac{1}{2},1,\frac{3}{2},2,\frac{5}{2},...$
octupole	3	1	0	tune spread: m=0
octupole	3	-1	2	$\frac{m}{2}$: $\frac{1}{2},1,\frac{3}{2},2,\frac{5}{2},...$
octupole	3	-3	4	$\frac{m}{4}$: $\frac{1}{4},\frac{1}{2},\frac{3}{4},1,\frac{5}{4},...$

Example: CESR, with Q_y=9.588. The operating tune lies between the second order resonance at 9.5=19/2 (m=19, k=-1), and the third order resonance at 9.667=29/3 (m=29, k=-2). The second order resonance will be driven by the term

$$-\left(\frac{a}{2}\right)\frac{Q}{2\pi}\int_0^C ds'\beta(s')\frac{b_1(s')}{B_0\rho}\cos[\Phi(s)-2\Phi(s')]$$

The third order resonance will be driven by

$$-\left(\frac{a}{2}\right)^2\frac{Q}{2\pi}\int_0^C ds'\beta(s')^{3/2}\frac{b_2(s')}{B_0\rho}\cos[\Phi(s)-3\Phi(s')]$$

Since CESR has approximate superperiodicity 2, both of these driving terms, having m odd, are suppressed by the ring symmetry.

Hence, any breaking of that symmetry by a field error will tend to strength these two nearby resonances.

Two-dimensional resonances

When we consider both transverse planes together, not only do we have possible resonances in both planes, but we also have the possibility of **coupling** the motion from one plane into the other. The general resonance conditions, including both planes together, can be written as

$$k_x Q_x + k_y Q_y = m$$

Here k_x and k_y are integers; the order of the resonance is $|k_x|+|k_y|$. m is a positive integer, related to the Fourier harmonic of the errors, as in 1 dimension. If either k_x or k_y is zero, we have a one-dimensional resonance. If k_x and k_y both have the same sign, the resonance is called a *sum resonance*. Such resonances are just as dangerous as one-dimensional resonances, and can cause beam loss. If k_x and k_y have opposite signs, then the resonance is called a *difference resonance*. Difference resonances represent conditions of energy exchange from one plane to another, and generally do not lead to beam loss. In electron machines requiring flat beams, however, these resonances will lead to an increase in the vertical beam dimension.

Example: third order resonances

$3Q_x = m$

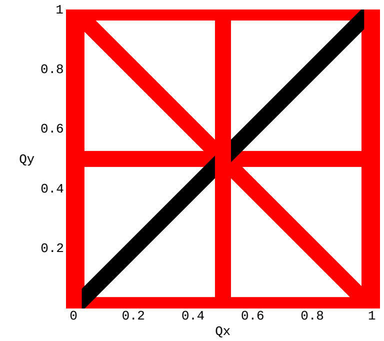
$2Q_x + Q_y = m \qquad 2Q_x - Q_y = m$

$Q_x + 2Q_y = m \qquad Q_x - 2Q_y = m$

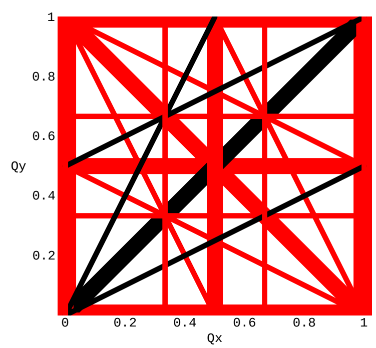
$3Q_y = m$

To keep track of all this, we usually use a graphical tool called a *working diagram*. This is a two dimensional plot of the vertical and horizontal tunes (*the tune plane*). Lines are drawn on this plot corresponding to the values of the tunes that satisfy the resonance conditions. One then plots the design machine tune on this diagram, and can immediately see how close the operating point is to resonance conditions.

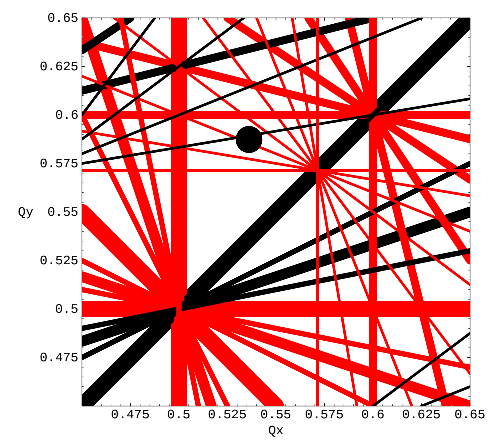
Examples: In these figures, red lines are sum resonances; black lines are difference resonances



First and second order resonance lines



First, second and third order resonance lines



Resonance lines to 7th order; CESR tunes are shown as a dot