LECTURE 15

Non-linear transverse motion

Phase-amplitude variables

Second –order (quadrupole-driven) linear resonances

Third-order (sextupole-driven) non-linear resonances

Phase-amplitude variables

Although the perturbation approach discussed in the previous lecture allows a general discussion of the conditions for resonance, to analyze the motion in phase space near a resonance in detail, we have to go back to the full equation of motion:

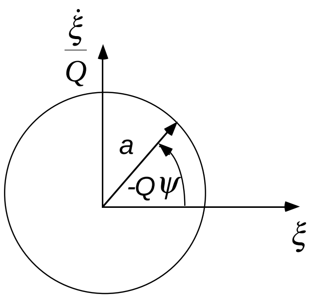
Lecture 16, p. 16:

$$\ddot{\xi} + Q^2\xi = -Q^2\beta^{3/2}\frac{\Delta B}{B_0\rho}$$

with the driving term given by

$$-Q^2\beta^{3/2}\frac{\Delta B}{B_0\rho} = -Q^2\sum_{m=-\infty}^{\infty} C_{m,n}\exp[im\psi]\xi(\psi)^n$$

To solve this equation, we have to make two more changes of the *phase space* variables. The phase space variables associated with the Floquet coordinates are $(\xi, \frac{\dot{\xi}}{Q})$. For purely linear motion, the phase space is a circle:



We will change to *polar co-ordinates* (r, ϕ) in phase space, where

$$\phi = -\tan^{-1}\frac{\dot{\xi}}{Q\xi}$$

$$r^2 = \xi^2 + \left(\frac{\dot{\xi}}{Q}\right)^2$$

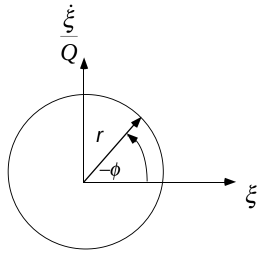
These are sometimes called “phase-amplitude” variables, because for purely linear motion

$$r = a$$

the invariant amplitude of the motion, and

$$\phi = -\tan^{-1}\frac{-Qa\sin Q\psi}{Qa\cos Q\psi} = Q\psi = \Phi$$

the betatron phase.



A particle executing purely linear motion has constant r, and ϕ advances by $2\pi Q$ every revolution. The conversion from (r,ϕ) to Floquet coordinates is

$$\xi = r\cos\phi \quad \dot{\xi} = -Qr\sin\phi$$

We will now proceed to transform the equation of motion into phase-amplitude variables. Then we will identify the resonance

driving terms, and ignore all other terms. When we are done, we will have an equation that we can integrate to get the trajectories in phase space.

For the radial coordinate, we have

$$r^2 = \xi^2 + \left(\frac{\dot{\xi}}{Q}\right)^2 \Rightarrow \frac{dr^2}{d\psi} = 2\dot{\xi}\xi + 2\dot{\xi}\frac{\ddot{\xi}}{Q^2} = 2\dot{\xi}\left(\xi + \frac{\ddot{\xi}}{Q^2}\right)$$

$$= 2\dot{\xi}\left(\xi + \frac{1}{Q^2}\left(-Q^2\xi - Q^2\sum_{m=-\infty}^{\infty} C_{m,n}\exp[im\psi]\xi(\psi)^n\right)\right)$$

$$= -2\dot{\xi}\sum_{m=-\infty}^{\infty} C_{m,n}\exp[im\psi]\xi(\psi)^n$$

Eliminating the Floquet coordinates gives

$$\frac{dr^2}{d\psi} = 2Q\cos^n\phi\sin\phi\, r^{n+1}\sum_{m=-\infty}^{\infty} C_{m,n}\exp[im\psi]$$

For the polar angle variable, we have

$$\tan\phi = -\frac{\dot{\xi}}{Q\xi} \Rightarrow \frac{d\phi}{d\psi} = Q\left[1 + \frac{Q^2\xi^{n+1}\sum_{m=-\infty}^{\infty} C_{m,n}\exp[im\psi]}{Q^2\xi^2 + \dot{\xi}^2}\right]$$

$$= Q\left[1 + \cos^{n+1}\phi\, r^{n-1}\sum_{m=-\infty}^{\infty} C_{m,n}\exp[im\psi]\right]$$

We'll now specialize to a particular type of field error. We'll start with quadrupole errors, for which the motion remains linear.

Second-order (quadrupole-driven) resonances

A quadrupole-driven resonance corresponds to n=1. The equations of motion are

$$\frac{dr^2}{d\psi} = 2Qr^2\cos\phi\sin\phi\sum_{m=-\infty}^{\infty} C_{m,1}\exp[im\psi]$$

$$\frac{d\phi}{d\psi} = Q\left[1 + \cos^2\phi\sum_{m=-\infty}^{\infty} C_{m,1}\exp[im\psi]\right]$$

For a single resonance, only one value of m will be important. For that value of m, we have

$$C_{m,1}\exp[im\psi] = \frac{1}{2\pi Q}\int_0^C ds'\beta(s')\frac{b_1(s')}{B_0\rho}\exp[im(\psi - \psi')]$$

Combining the positive and negative values of m gives

$$C_{m,1}\exp[im\psi]+C_{-m,1}\exp[-im\psi]=$$

$$\frac{1}{\pi Q}\int_0^C ds'\beta(s')\frac{b_1(s')}{B_0\rho}\cos[m(\psi-\psi')]$$

$$=\frac{1}{\pi Q}\left(A_{m1}\cos m\psi+B_{m1}\sin m\psi\right)$$

$$C_{0,1}=\frac{A_{01}}{2\pi Q}$$

in which

$$A_{m1}=\int_0^C ds'\beta(s')\frac{\Delta B'(s')}{B_0\rho}\cos\left[\frac{m}{Q}\Phi(s')\right]$$

$$B_{m1}=\int_0^C ds'\beta(s')\frac{\Delta B'(s')}{B_0\rho}\sin\left[\frac{m}{Q}\Phi(s')\right]$$

These are the harmonic coefficients that will drive the resonance. The equations of motion become

$$\frac{dr^2}{d\psi}=\frac{2}{\pi}r^2\cos\phi\sin\phi\left(A_{m1}\cos m\psi+B_{m1}\sin m\psi\right)$$

$$\frac{d\phi}{d\psi}=Q+\frac{1}{\pi}\cos^2\phi\left(A_{m1}\cos m\psi+B_{m1}\sin m\psi\right)$$

We expand out the trig functions:

$$\frac{dr^2}{d\psi}=\frac{r^2}{2\pi}\left[\begin{array}{l}A_{m1}\left(\sin(2\phi-m\psi)+\sin(2\phi+m\psi)\right)\\+B_{m1}\left(\cos(2\phi-m\psi)-\cos(2\phi+m\psi)\right)\end{array}\right]$$

Recall that a quadrupole can only drive a second order resonance: this is reflected in the term with argument $2\phi-m\psi\approx 2\left(Q-\frac{m}{2}\right)\psi$, which drives the second order resonance at $Q\approx\frac{m}{2}$. (The terms with arguments $2\phi+m\psi$ do not drive any resonances, since Q is always positive: they correspond to rapidly oscillating terms that may be neglected).

So we have

$$\frac{dr^2}{d\psi}\cong\frac{1}{2\pi}r^2\left[A_{m1}\sin(2\phi-m\psi)+B_{m1}\cos(2\phi-m\psi)\right]$$

A similar treatment of the equation for ϕ gives

$$\frac{d\phi}{d\psi}\cong Q+\frac{A_{01}}{4\pi}+\frac{A_{m1}}{2\pi}\cos m\psi+\frac{B_{m1}}{2\pi}\sin m\psi$$
$$+\frac{1}{4\pi}\left[A_{m1}\cos(2\phi-m\psi)-B_{m1}\sin(2\phi-m\psi)\right]$$

The terms with $\cos m\psi$ and $\sin m\psi$ will oscillate rapidly, and can be neglected. The m=0 term corresponds to the quadrupole-induced *tune shift*:

$$\frac{d\phi}{d\psi}\cong\left(Q+\frac{A_{01}}{4\pi}\right)+\frac{1}{4\pi}\left[A_{m1}\cos(2\phi-m\psi)-B_{m1}\sin(2\phi-m\psi)\right]$$

The tune shift is

$$\Delta Q = \frac{A_{01}}{4\pi} = \frac{1}{4\pi}\int_0^C ds' \beta(s') \frac{\Delta B'(s')}{B_0 \rho}$$

which we can recognize from our previous work (Lecture 8, p 19)

We need to make one more manipulation: we can simplify the arguments of the trig functions by introducing the angle

$$\phi' = \phi - \frac{m\psi}{2}$$

Then the two equations for phase and amplitude become

$$\frac{d\phi'}{d\psi} = \frac{d\phi}{d\psi} - \frac{m}{2} = (Q+\Delta Q) - \frac{m}{2} + \frac{1}{4\pi}\left[A_{m1}\cos 2\phi' - B_{m1}\sin 2\phi'\right]$$

$$\frac{dr^2}{d\psi} = \frac{1}{2\pi} r^2 \left[A_{m1}\sin 2\phi' + B_{m1}\cos 2\phi'\right]$$

Combining these equations gives us a differential equation for the *phase space trajectories*, that is, an equation for r as a function of ϕ'

$$\frac{dr^2}{d\phi'} = \frac{dr^2}{d\psi}\left(\frac{d\phi'}{d\psi}\right)^{-1} = \frac{\frac{r^2}{2\pi}\left[A_{m1}\sin 2\phi' + B_{m1}\cos 2\phi'\right]}{(Q+\Delta Q) - \frac{m}{2} + \frac{1}{4\pi}\left[A_{m1}\cos 2\phi' - B_{m1}\sin 2\phi'\right]}$$

$$\frac{dr}{r} = \frac{\frac{1}{2}\frac{d(2\phi')}{4\pi}\left[A_{m1}\sin 2\phi' + B_{m1}\cos 2\phi'\right]}{(Q+\Delta Q) - \frac{m}{2} + \frac{1}{4\pi}\left[A_{m1}\cos 2\phi' - B_{m1}\sin 2\phi'\right]}$$

This equation can be integrated relatively easily. The result is

$$r^2 = \frac{a^2}{1 + \frac{1}{4\pi}\frac{A_{m1}\cos 2\phi' - B_{m1}\sin 2\phi'}{Q+\Delta Q - \frac{m}{2}}}$$

$$= \frac{a^2}{1 + \frac{1}{4\pi}\frac{A_{m1}\cos(2\phi - m\psi) - B_{m1}\sin(2\phi - m\psi)}{Q+\Delta Q - \frac{m}{2}}}$$

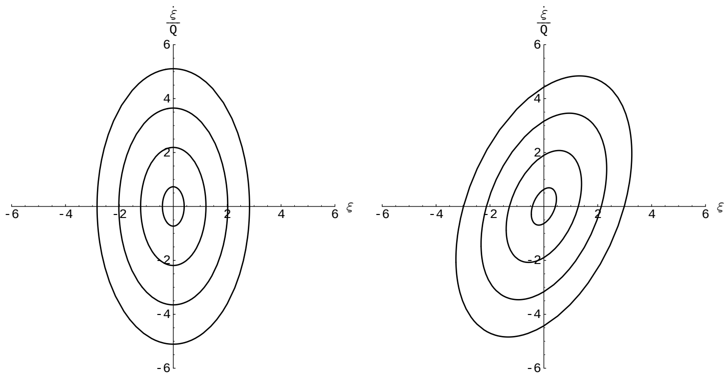
where a is a constant of integration; it can be interpreted as the value of r^2 far from the resonance, when the denominator of the resonant term $Q + \Delta Q - \frac{m}{2}$ is large.

To understand this result, we simplify it by taking B_{m1}=0, and m=1. Then, if we let $\delta Q = Q + \Delta Q - \frac{m}{2}$,

$$r^2 = \frac{a^2}{1 + \frac{A_{m1}}{4\pi\delta Q}\cos(2\phi - \psi)}$$

This is a family of ellipses in Floquet coordinate phase space, for various values of a. Let's plot some of these, for δQ=0.001,

A_{m2}=0.02, and for a ranging from 0.5 to 3.5. The left figure is for ψ=0, the right for ψ=π/4.



All the phase space trajectories are elliptical, even for the smallest value of a. The elliptical shape reflects the fact that a quadrupole

perturbation changes the lattice functions, and hence changes the Floquet transformation. We could restore the circular shapes in phase space if we redid the Floquet transformation, but used the new values of the lattice functions, after the introduction of the quadrupole errors.

This motion is linear and stable for all amplitudes. However, if we inspect the equation for the phase spaces ellipses, we see that, for a physical solution valid for all ϕ, we must have

$$1+\frac{A_{m1}}{4\pi\delta Q}\cos(2\phi-\psi)\geq 0 \Rightarrow \left|\frac{A_{m1}}{4\pi\delta Q}\right|\leq 1$$

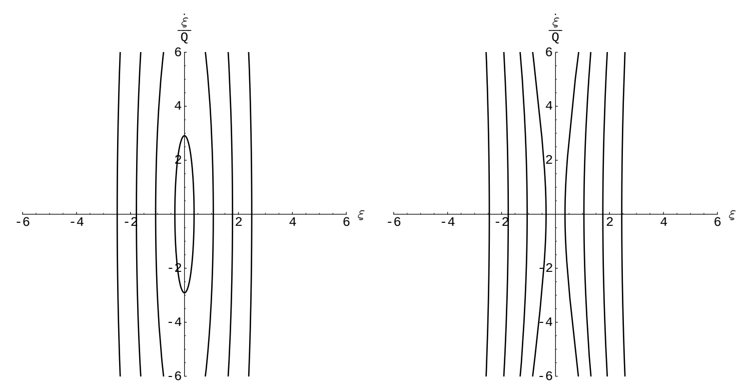
$$|\delta Q|\geq\frac{A_{m1}}{4\pi}=\frac{1}{4\pi}\int_0^C ds'\beta(s')\frac{\Delta B'(s')}{B_0\rho}\cos[2\Phi(s')]$$

This is the *second-order resonance stopband width*.

Note: if we had not assumed B_{m1}=0, we would have found for the stopband width

$$|\delta Q|\geq\frac{\sqrt{A_{m1}^2+B_{m1}^2}}{4\pi}$$

If the tune shift is larger than this value, then the motion is unstable *for all amplitudes*: The next two figures show the phase space just outside the stopband (bounded (stable) motion, left figure) and just inside the stopband (unbounded (unstable) motion, right figure).



Third-order (sextupole-driven) resonances

A sextupole-driven resonance corresponds to n=2. The equations of motion are

$$\frac{dr^2}{d\psi} = 2Qr^3 \cos^2\phi \sin\phi \sum_{m=-\infty}^{\infty} C_{m,2} \exp[im\psi]$$

$$\frac{d\phi}{d\psi} = Q\left[1 + r\cos^3\phi \sum_{m=-\infty}^{\infty} C_{m,2} \exp[im\psi]\right]$$

For a single resonance, only one value of m will be important. For that value of m, we have

$$C_{m,2} \exp[im\psi] = \frac{1}{2\pi Q}\int_0^C ds'\beta(s')^{3/2}\frac{b_2(s')}{B_0\rho}\exp[im(\psi-\psi')]$$

Combining the positive and negative values of m gives

$$C_{m,2}\exp[im\psi] + C_{-m,2}\exp[-im\psi] =$$

$$\frac{1}{\pi Q}\int_0^C ds'\beta(s')^{3/2}\frac{b_2(s')}{B_0\rho}\cos[m(\psi-\psi')]$$

$$= \frac{1}{\pi Q}\left(A_{m2}\cos m\psi + B_{m2}\sin m\psi\right)$$

in which

$$A_{m2} = \int_0^C ds'\beta(s')^{3/2}\frac{\Delta B''(s')}{2B_0\rho}\cos\left[\frac{m}{Q}\Phi(s')\right]$$

$$B_{m2} = \int_0^C ds'\beta(s')^{3/2}\frac{\Delta B''(s')}{2B_0\rho}\sin\left[\frac{m}{Q}\Phi(s')\right]$$

These are the harmonic coefficients that will drive the resonance. The equations of motion become

$$\frac{dr^2}{d\psi} = \frac{2}{\pi}r^3\cos^2\phi\sin\phi\left(A_{m2}\cos m\psi + B_{m2}\sin m\psi\right)$$

$$\frac{d\phi}{d\psi} = Q + \frac{1}{\pi}r\cos^3\phi\left(A_{m2}\cos m\psi + B_{m2}\sin m\psi\right)$$

We expand out the trig functions:

$$\frac{dr^2}{d\psi} = \frac{1}{4\pi}r^3\left[\begin{array}{l} A_{m2}\begin{pmatrix}\sin(\phi+m\psi)+\sin(3\phi+m\psi) \\ +\sin(\phi-m\psi)+\sin(3\phi-m\psi)\end{pmatrix} \\ -B_{m2}\begin{pmatrix}\cos(\phi+m\psi)+\cos(3\phi+m\psi) \\ -\cos(\phi-m\psi)-\cos(3\phi-m\psi)\end{pmatrix}\end{array}\right]$$

The various terms in the expansion correspond to different resonance orders. Recall that a sextupole can drive first order or third order resonances: this is reflected in the term with argument $3\phi - m\psi \approx 3\left(Q - \frac{m}{3}\right)\psi$, which drives the third order resonance at $Q \approx \frac{m}{3}$, and the term with argument $\phi - m\psi \approx (Q-m)\psi$, which drives the first order resonance at $Q \approx m$. (The terms with arguments $3\phi + m\psi$ and $\phi + m\psi$ do not drive any resonances, since Q is always positive: they correspond to rapidly oscillating terms that may be neglected).

Since we are only interested in the terms that drive the third order resonance, we have

$$\frac{dr^2}{d\psi} \cong \frac{1}{4\pi} r^3 \left[A_{m2} \sin(3\phi - m\psi) + B_{m2} \cos(3\phi - m\psi) \right]$$

A similar treatment of the equation for ϕ gives

$$\frac{d\phi}{d\psi} \cong Q + \frac{1}{8\pi} r \left[A_{m2} \cos(3\phi - m\psi) - B_{m2} \sin(3\phi - m\psi) \right]$$

As before, we simplify the arguments of the trig functions by introducing the angle

$$\phi' = \phi - \frac{m\psi}{3}$$

Then the two equations for phase and amplitude become

$$\frac{d\phi'}{d\psi} = \frac{d\phi}{d\psi} - \frac{m}{3} = Q - \frac{m}{3} + \frac{1}{8\pi} r \left[A_{m2} \cos 3\phi' - B_{m2} \sin 3\phi' \right]$$

$$\frac{dr^2}{d\psi} = \frac{1}{4\pi} r^3 \left[A_{m2} \sin 3\phi' + B_{m2} \cos 3\phi' \right]$$

Combining these equations gives us a differential equation for the *phase space trajectories*, that is, an equation for r as a function of ϕ'

$$\frac{dr^2}{d\phi'} = \frac{dr^2}{d\psi} \left(\frac{d\phi'}{d\psi} \right)^{-1} = \frac{\frac{r^3}{4\pi} \left[A_{m2} \sin 3\phi' + B_{m2} \cos 3\phi' \right]}{Q - \frac{m}{3} + \frac{r}{8\pi} \left[A_{m2} \cos 3\phi' - B_{m2} \sin 3\phi' \right]}$$

This equation can be integrated! The result gives the phase space trajectories in the vicinity of a third-order resonance:

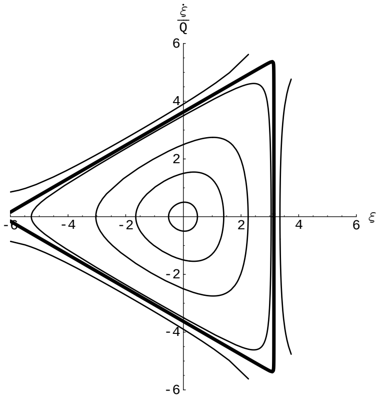
$$a^2 = r^2 + r^3 \frac{A_{m2} \cos 3\phi' - B_{m2} \sin 3\phi'}{12\pi \left(Q - \frac{m}{3} \right)}$$

$$= r^2 + r^3 \frac{A_{m2} \cos(3\phi - m\psi) - B_{m2} \sin(3\phi - m\psi)}{12\pi \left(Q - \frac{m}{3} \right)}$$

where a is a constant of integration; it can be interpreted as the value of the invariant far from the resonance, when the denominator of the resonant term $Q - \frac{m}{3}$ is large. To understand this result, we simplify it by taking B_{m2}=0, and look at the point in the ring where ψ=0. Then, if we let $\delta Q = Q - \frac{m}{3}$,

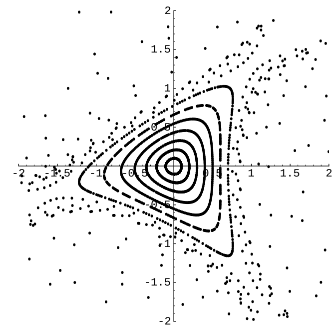
$$a^2 = r^2 + r^3 \frac{A_{m2} \cos 3\phi}{12\pi \delta Q}$$

This is a family of curves in Floquet coordinate phase space, for various values of a. Let's plot some of these, for δQ=0.001, A_{m2}=0.004, and for a ranging from 0.5 to 3.9

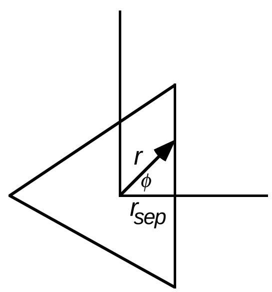


The circular trajectories in the center correspond to linear motion with small *a*. As the amplitude of the motion *a* increases, the trajectories distort from circular into triangular shapes, characteristic of a third-order resonance. The separatrix, the

boundary of stable motion, is the heavy triangular line. Motion within the separatrix is stable. Just outside the separatrix, the motion tends to be *chaotic:* that is, small changes in the initial conditions for the motion can lead to large changes after many turns. Numerical turn-by turn simulation of motion near the third-order resonance:



The corners of the triangle are the three *fixed points*.



The radial distance r_{sep} to the separatrix is a measure of the maximum amplitude of stable motion. From geometry, on the vertical separatrix, we have

$r = \dfrac{r_{sep}}{\cos\phi}$ and $a_{sep}^2 = r^2 + r^3 \dfrac{A_{m2}\cos 3\phi}{12\pi\delta Q}$, where a_{sep} is the value of *a* corresponding to the separatrix. So,

$$a_{sep}^2 \cos^3\phi = r_{sep}^2 \cos\phi + r_{sep}^3 \frac{A_{m2}\cos 3\phi}{12\pi\delta Q}$$

Equating the coefficients of cos ϕ and cos 3ϕ gives

$$a_{sep} = \frac{8\pi\delta Q}{\sqrt{3}A_{m2}}$$

Since a^2 corresponds to the emittance of the particle, we have an expression for the *third-order resonance width* for a particle of emittance ε:

$$2\delta Q = \sqrt{3\varepsilon}\,\frac{A_{m2}}{4\pi} = \frac{\sqrt{3\varepsilon}}{4\pi}\int_0^C ds'\,\beta(s')^{3/2}\,\frac{B''(s')}{2B_0\rho}\cos[3\Phi(s')]$$

Note: we ignored the B_{m2} coefficient, for simplicity. Including this coefficient, the resonance width is given by

$$2\delta Q = \sqrt{3\varepsilon}\,\frac{\sqrt{A_{m2}^2 + B_{m2}^2}}{4\pi}.$$

The resonance widths may be controlled through the azimuthal distribution of the sextupoles. With two families of sextupoles at appropriate locations, both the A_{m2} and B_{m2} coefficients may be minimized.

Example:

500 m model accelerator, with FODO lattice. From Lecture 8, p 37, we saw that we could compensate the natural chromaticity by placing two sextupoles in the lattice: a sextupole of strength m_D=-105 m^{-3} at any D quad, where $\beta_{x,D}$ = 4.8 m, and a sextupole of strength m_F = 59 m^{-3} at the adjacent F quad, where $\beta_{x,F}$ = 16.8 m. What is the third-order resonance width produced by these sextupoles, for a beam of emittance ε=10^{-6} m-rad?

The sextupoles had length L_s = 0.1 m. Using $m = \dfrac{B''}{B_0\rho}$, the resonance width in the x-plane is

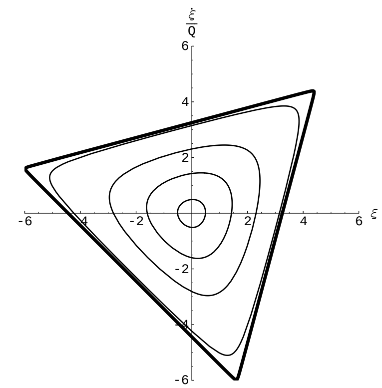
$$2\delta Q_x = \frac{\sqrt{3\varepsilon}}{4\pi}\frac{L_s}{2}\left[\beta_{x,F}^{\frac{3}{2}} m_F + \beta_{x,D}^{\frac{3}{2}} m_D \cos\left(\frac{3\mu}{2}\right)\right]$$

where μ=1.178 is the phase advance per cell.

Plugging in the numbers gives $2\delta Q_x = 0.029$.

Note: this is an underestimate, since we have ignored the B_{m2} coefficient.

As for the second order resonance, the orientation of the phase space orbits for a third order resonance in Floquet coordinates depends on the azimuthal location at which the particles are observed. The following figure shows phase space at ψ=π/4 instead of ψ=0.



In *resonant extraction*, the region of stable phase space is gradually driven to zero, typically by increasing the strength of the nonlinear fields driving the resonance. This causes all particles eventually to flow along the separatrices. A magnetic or electrostatic septum is placed at an appropriate azimuth to intercept the particles flowing along the separatrix, and they are diverted into a magnetic extraction channel.